

# Two-way Networks: when Adaptation is Useless

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## Abstract

Most wireless communication networks are two-way, where nodes act as both sources and destinations of messages. This allows for “adaptation” at or “interaction” between the nodes – a node’s channel inputs may be functions of its message(s) and previously received signals, in contrast to feedback-free one-way channels where inputs are functions of messages only. How to best adapt, or cooperate, is key to two-way communication, rendering it complex and challenging. However, examples exist of channels where adaptation is *not* beneficial from a capacity perspective; it is known that for the point-to-point two-way modulo 2 adder and Gaussian channels, adaptation does not increase capacity. We ask whether analogous results hold for several multi-user two-way networks.

We first consider deterministic two-way channel models: the binary modulo-2 addition channel, the “deterministic, invertible and cardinality constrained” model, and the linear deterministic channel model which models Gaussian channels at high SNR. For these deterministic models we obtain the capacity region for the two-way multiple access/broadcast channel, the two-way Z channel and the two-way interference channel (under certain “partial” adaptation constraints in some regimes). In all cases we permit all nodes to adapt their channel inputs to past outputs (except for portions of the linear high-SNR two-way interference channel where we only permit 2 of the 4 nodes to fully adapt). However, we show that this adaptation is useless from a capacity region perspective. That is, the two-way fully or partially adaptive capacity region consists of two parallel “one-way” regions operating simultaneously in opposite directions, achieved by strategies where the channel inputs at each use do not adapt to previous inputs. Finally, we consider the noisy Gaussian two-way interference channel, and show that partial adaptation is useless when the interference is very strong. In the strong and weak interference regimes, we show that the non-adaptive Han and Kobayashi scheme utilized in parallel in both directions achieves to within a constant gap for the symmetric rate of the fully (for some regimes) or partially (for the remaining regimes) adaptive models.

The central technical contribution is the derivation of new, computable outer bounds which allow for adaptation (or partial adaptation in some interference channel regimes). Inner bounds follow from known, non-adaptive achievability schemes of the corresponding one-way channel models.

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## I. INTRODUCTION

Two-way communication, where clients A and B wish to exchange a stream of information, is a natural form of communication of great relevance in present and future wireless networks. Applications include two-way high data-rate tele-medicine over wireless broadband links, mobile video conferencing over next generation cellular networks such as 3GPP LTE and IEEE 802.16, the synchronization of data among terminals, and communication between a base station and clients. Indeed, much of our current wireless communication is already two-way in nature, but it is not treated as such in practice. Rather, current channel coding schemes orthogonalize the two directions, rendering the two-way channel equivalent to two one-way communication links. While this is simple to implement, whether such non-adaptive two-way coding schemes are optimal from a capacity perspective remains an open question.

What makes two-way communications, in which two (or more) users exchange messages over the same shared channel, challenging are the possibilities that stem from having nodes be both sources and destinations of messages. This permits them to adapt their channel inputs to their past received signals. Such two-way adaptation was first considered in the point-to-point two-way channel by Shannon [3]. Shannon's inner and outer bounds [3] are not tight in general, and a general computable<sup>1</sup> formula for the capacity region of the point-to-point two-way channel remains open.

However, encouragingly, capacity is known for several point-to-point two way channel models where the interaction between ones own signal and that of the other user may be resolved. For example, in the two-way modulo 2 binary adder channel where channel outputs  $Y_1 = Y_2 = X_1 \oplus X_2$  for binary inputs  $X_1, X_2$  and  $\oplus$  modulo 2 addition, the capacity region is one bit per user per channel use, as each user is able to “undo” the effect of the other (as shown in Fig. 1 (a)), something that is not possible (at least not in one channel use) for the elusive binary multiplier channel with  $Y_1 = Y_2 = X_1 X_2$ . In the binary modulo 2 adder channel, information independently flows in the  $\rightarrow$  and the  $\leftarrow$  “directions” and nodes need not interact, or adapt their current inputs to past outputs, to achieve capacity. In a similar fashion, the capacity of a two-way Gaussian point-to-point channel is equal to two parallel Gaussian channels (as shown in Fig. 1 (b)), which may be achieved without the use of adaptation at the nodes [5].

**A note on terminology.** In this work, “adaptation” or “interaction” is said to take place when the next channel input of a node is a non-trivial function of that node's past received signals. One may alternatively use the terms “feedback” or “cooperation” instead of adaptation or interaction. However, we feel that “adaptation” and “interaction” better highlights the nature of *two-way* communications where there is no real notion of feedback as all links may carry information for forward and backwards directions

<sup>1</sup>By computable we mean single-letter expression without the use of unbounded cardinality auxiliary random variables. Multi-letter formulas for the capacity of two-way channels exist, see the expressions involving directed information over code-trees of [4].

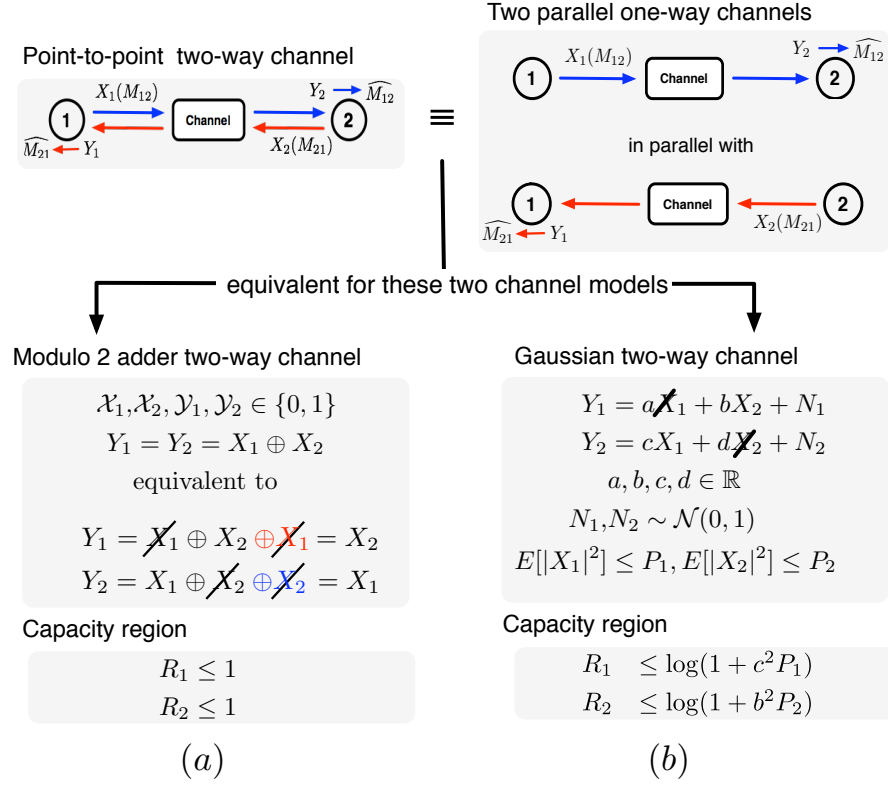


Fig. 1. Examples of point-to-point two-way channels breaking up into parallel channels.

simultaneously. That is, the term “feedback” suggests that the backwards or cooperative links serve to improve communication in the forward direction, while in two-way channels, each link can carry a combination of forward and backward traveling messages. “Cooperation” may also be used and more accurately reflects the fact that nodes may help each other in multi-user two-way channels. However, again – cooperation is a term which has been used in many existing one-way communication scenarios. We feel fresh terminology to emphasize the fact that all nodes may adapt their transmission to each other is useful, as it does not contain any notion of directionality.

### A. Contributions

We are interested in determining whether there exist examples of multi-user two-way channels rather than point-to-point two-way channels where, even though nodes may adapt current inputs to past outputs, this is *not* beneficial from a capacity region perspective. In two-way networks, one may expect adaptation to, in general, be useful and enlarge the capacity region. For example, in multi-user Gaussian channels one may intuitively expect adaptation to allow for correlation between channel inputs which may translate to coherent gains. However, as we will see, there exist multi-user channels (where coherent gains are impossible) for which adaptation is useless. In particular, we introduce three two-way channel models:

- 1) the **two-way Multiple Access / Broadcast channel (MAC/BC)** in which there are 4 messages and 3 terminals forming a MAC channel in the  $\rightarrow$  direction (2 messages) and a BC channel in the opposite  $\leftarrow$  direction (2 messages);
- 2) the **two-way Z channel** in which there are 6 messages and 4 terminals forming a Z channel in the  $\rightarrow$  direction (3 messages) and another Z channel in the opposite  $\leftarrow$  direction (3 messages);
- 3) the **two-way interference channel (IC)** with 4 messages and 4 terminals forming an IC in the  $\rightarrow$  direction (2 messages) and another IC in the  $\leftarrow$  direction (2 messages).

We emphasize that all nodes are permitted to adapt, i.e. channel inputs at node  $j$  at time  $i$  may be functions of the received signals at node  $j$  from times 1 to  $i - 1$ , and that data and “feedback” share the same links, i.e. there are no orthogonal feedback links anywhere in these networks. For these channel models, our central contributions may be summarized as follows:

- We consider **deterministic modulo 2 adder channels** for each of the three above channel models. These are the simplest examples of multi-user two-way channels where one might intuitively expect adaptation to be useless. All inputs and outputs are binary, there is no noise, and the signals are added modulo 2. For these channel models, and slight generalizations thereof, we obtain outer bounds, and demonstrate that non-adaptive time-sharing schemes between nodes transmitting in the same “direction” achieve capacity. Nodes transmitting data in the opposite directions may simultaneously transmit as in the point-to-point modulo 2 adder and Gaussian channel models.
- We next consider **linear deterministic models** of the three two-way channels above which model Gaussian channels at high SNR [6] and again ask whether adaptation may increase the capacity regions beyond that of two parallel one-way multi-user channels in the  $\rightarrow$  and  $\leftarrow$  directions. We will show that it does not for the first two channel models by obtaining their capacity regions. For the two-way interference channel, we will show that *partial adaptation* where only two of the four nodes may adapt, can “block” the two-way information flow and destroy the ability to relay / cooperate. Under this constraint, we obtain the capacity region which is equal to two non-adaptive interference channels. In addition, in some regimes of the relative link strengths, we obtain the capacity region for symmetric model with *full adaptation* where all four nodes are permitted to adapt. The central contribution here is the derivation of the capacity region which allows for fully (or partially for the two-way IC) adaptive channel models; typically two-way problems of this form end up having multi-letter expressions or auxiliary random variables.
- Finally, we consider one example of a Gaussian channel: the **symmetric two-way Gaussian IC** where all “direct” links are equal and all “cross-over” links are equal. We derive new, computable outer bounds for the symmetric sum-rates for this Gaussian channel model and show that: a) adaptation is useless in very strong interference for the partially adaptive model, b) in strong but not very strong interference, non-

adaptive schemes perform to within 1 bit per user per direction of the fully adaptive capacity region, and c) the particular non-adaptive Han and Kobayashi scheme of [7] employed in each direction, achieves to within a constant gap (2 bits per user per direction maximally) of fully or partially adaptive outer bounds in all other regimes. In general, when all nodes are permitted to adapt, we do not believe that a non-adaptive scheme will achieve to within a constant gap for all regimes but this is left open. The emphasis of this work is on demonstrating when *adaptive* schemes are useless, and when, even if adaptation is permitted, it does not significantly increase the capacity region.

### B. Related Work

This work builds on three main bodies of literature: point-to-point two-way channels, one-way multi-user deterministic channels, and one-way multi-user channels with feedback. Little work exists thus far on two-way multi-user channels.

The capacity region of the general point-to-point discrete memoryless two-way channel may be written in terms of the limit of multi-letter expressions as in [3, Section 15], or [4, Theorem 4.1]. Given the complexity in computing this capacity region, it is not entirely satisfying and the capacity region of the two-way channel is still generally considered to be open. The binary multiplier channel (BMC) [8]–[12] is a nice example of a deterministic, binary, common output two-way channel where capacity is not exactly known, though its capacity may be expressed in terms of directed information as in [4, Corollary 4.1]. However, the capacity regions of several particular two-way channels shown in Fig. 1 are known; in both examples adaptation is useless and the capacity region decomposes into two parallel one-way channels. These models were the inspiration for asking whether such examples exist in multi-user two-way networks.

The first of our three channel models is a two-way MAC/BC channel. The capacity regions of the linear-deterministic one-way MAC and BC channels were obtained in [13]. An achievable rate region and an outer bound of a similar two-way and adaptive multi-user half-duplex two-way channel is derived in [14], [15] for Gaussian and discrete memoryless channels (DMC), respectively. In particular, the achievable rate region derived employs adaptation using Block Markov encoding, and the outer bound contains both auxiliary random variables and messages in its expression and is thus difficult to compute. These works differ from our model in that we assume full-duplex operation, have 2 broadcast messages rather than a common one (and hence 4 total messages rather than 3), and we will be considering deterministic channel models rather than general DMC and Gaussian models. Other than [14], [15], the two-way MAC/BC has not been considered, and bears most resemblance to a combined MAC channel with feedback and BC channel with feedback (see references in [16, Ch. 17, Bibliographic Notes]), though we note that in our two-way model there are no “free” feedback links – any feedback must travel over the same links

as the data in the opposite direction, and hence the MAC and BC with feedback results are not directly applicable.

The second channel model we consider is the two-way Z channel, with 6 messages. The one-way Z channel (with 3 messages, rather than the Z Interference channel with 2 messages) was first studied in [17], in which the capacity region of a special class of degraded Z channels and an outer bound for the general Z channel are obtained. The capacity region of the one-way deterministic Z channel with invertability constraints similar in flavor to those for the class of deterministic interference channels for which capacity is known [18], is found in [19], which will be of use here.

The last channel model considered is the two-way linear deterministic IC in which there are 4 messages and 4 terminals forming interference channels in the  $\rightarrow$  and  $\leftarrow$  directions. The capacity region of the one-way modulo 2 adder IC is known [16] and is a special example of a more general class of deterministic IC for which capacity is known [18], including the one-way linear deterministic IC [13]. The work here is also related to one-way ICs with perfect output feedback [20], [21], with rate-limited feedback [22], with generalized feedback [23], and interfering feedback [20], [24]. In all these channel models only two messages are present and the “feedback” links, whether perfect, noisy, or interfering still serve only to further rates in the forward direction. The tradeoff between sending new information versus feedback on each of the links is not addressed. The only other example of such a 4-message two-way interference channel besides our prior work [1], [2], [25] is in Section VI of [24], where an example of a linear deterministic scheme in a specific regime is provided which shows that, at least for one particular asymmetric linear deterministic two-way IC with weak interference in the  $\rightarrow$  and strong interference in the  $\leftarrow$  direction, that adaptation can significantly improve the capacity region over non-interaction. The general capacity region of the linear deterministic two-way IC (with 4 messages) remains open in general despite the example in [24] and the work here. However, in this work we will show capacity for *symmetric* channels with partial adaptation and full adaptation in certain regimes. One final word on terminology: we will refer to the 4 message two-way IC as the “two-way IC” and the 2 message channel of [24] – considered in all sections but Section VI – as the “two-way interference channel with interfering feedback” to emphasize that the rates are still flowing in one direction only. Further comparisons and relationships with ICs with/without feedback [7], [21], [24], [26] will be made in Section V and VI.

### C. Outline

We will first introduce our notation, definitions of adaptation and capacity regions, and the three general channel models considered in Section II. We then proceed to explore each of these three channel models separately. In Section III we consider the two-way MAC/BC channel and show that adaptation is useless

for three deterministic channel models: 1) the binary modulo 2 adder channel, 2) a generalization of this which we term the “deterministic, invertible and cardinality constrained” model, and finally 3) the linear deterministic channel. In all three channels time-sharing is shown to achieve capacity. In Section IV we again show that adaptation is useless for the same three deterministic models as in the MAC/BC (but now for the two-way Z channel). In Section V we move on to the deterministic IC. For the 1) binary modulo 2 channel we show that adaptation is useless, and show a similar result for its generalization 2) the “deterministic, invertible and cardinality constrained” model. For the 3) linear deterministic model we show that adaptation is useless for  $2/3 < \alpha$  (where  $\alpha$  denotes the ratio of cross-over to direct links, as in [7]), and show that partial adaptation is useless for the remaining channel conditions. We also obtain the general asymmetric capacity region for the linear deterministic channel model under partial adaptation, which is equal to two parallel ICs in opposite directions. Finally, in Section VI we consider the Gaussian two-way IC and show that a non-adaptive scheme achieves within a constant gap (and in one case capacity) of any partially (sometimes fully) adaptive scheme. We conclude in Section VII with some general observations and intuition as to when adaptation is useful, which may be extracted from these examples of two-way multi-user channel models.

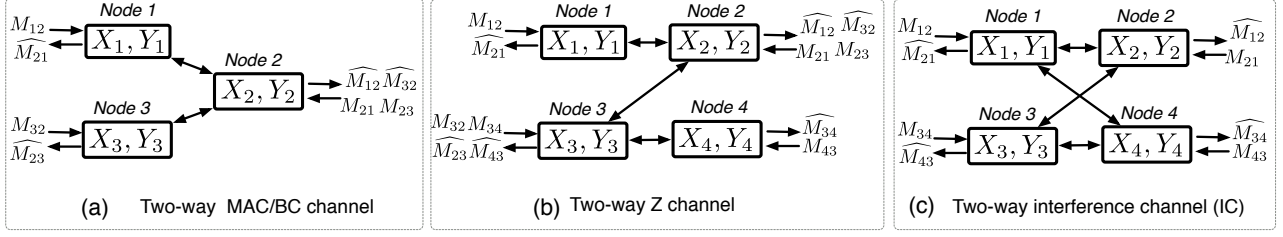
## II. MODELS, DEFINITIONS AND NOTATIONS

We consider three multi-user two-way channels, as shown in Fig. 2. All nodes act as both transmitters (encoders) and receivers (decoders), and we illustrate two specific channel models and per-channel use input-output equations that we will consider: the binary modulo 2 adder and the linear deterministic channel versions of all models. To summarize, we consider the following three models:

- *the two-way MAC/BC channel*: transmitters 1 and 3 send independent messages  $M_{12}$  and  $M_{32}$  to receiver 2, respectively, forming a Multiple Access Channel (MAC) in the  $\rightarrow$  direction. Meanwhile, transmitter 2 sends two independent messages  $M_{21}$  and  $M_{23}$  to receivers 1 and 3, respectively, forming a Broadcast Channel (BC) in the  $\leftarrow$  direction.
- *the two-way Z channel*: transmitters 1 and 4 send messages  $M_{12}$  and  $M_{43}$  to receivers 2 and 3 respectively. Transmitters 2 and 3 send messages  $(M_{21}, M_{23})$  and  $(M_{32}, M_{34})$  to receivers 1,3 and 2,4 respectively. We thus have two Z channels in opposite directions.
- *the two-way interference channel*: transmitters 1 and 3 send messages  $M_{12}$  and  $M_{34}$  to receivers 2 and 4, respectively, forming an IC in the  $\rightarrow$  direction. Similarly, transmitters 2 and 4 send messages  $M_{21}$  and  $M_{43}$  to receivers 1 and 3 respectively, forming another IC in the  $\leftarrow$  direction.

For each of these models, let  $M_{jk}$  denote the message from node  $j$  to node  $k$ ; all messages are independent and uniformly distributed over  $\mathcal{M}_{jk} := \{1, 2, \dots, 2^{nR_{jk}}\}$ , where the ranges of  $j, k$  depend on

$M_{jk}$  are independent messages from Tx  $j$  to Rx  $k$



**Binary modulo 2 adder**  $X_j \in \{0, 1\}$ ,  $\oplus$  is modulo 2 addition

$$Y_1 = X_1 \oplus X_2$$

$$Y_2 = X_1 \oplus X_2 \oplus X_3$$

$$Y_3 = X_2 \oplus X_3.$$

$$Y_1 = X_1 \oplus X_2$$

$$Y_2 = X_1 \oplus X_2 \oplus X_3$$

$$Y_3 = X_2 \oplus X_3 \oplus X_4$$

$$Y_4 = X_3 \oplus X_4.$$

$$Y_1 = X_1 \oplus X_2 \oplus X_4$$

$$Y_2 = X_1 \oplus X_2 \oplus X_3$$

$$Y_3 = X_2 \oplus X_3 \oplus X_4$$

$$Y_4 = X_1 \oplus X_3 \oplus X_4.$$

**Linear deterministic**  $X_j \in \{0, 1\}^N$ ,  $S = N \times N$  shift matrix,  $n_{jk}$  is # bit levels Tx  $j$  to Rx  $k$ ,  $N = \max(n_{jk})$ .

$$Y_1 = S^{N-n_{11}} X_1 + S^{N-n_{21}} X_2$$

$$Y_2 = S^{N-n_{12}} X_1 + S^{N-n_{22}} X_2 + S^{N-n_{32}} X_3$$

$$Y_3 = S^{N-n_{23}} X_2 + S^{N-n_{33}} X_3,$$

$$Y_1 = S^{N-n_{11}} X_1 + S^{N-n_{21}} X_2$$

$$Y_2 = S^{N-n_{12}} X_1 + S^{N-n_{22}} X_2 + S^{N-n_{32}} X_3$$

$$Y_3 = S^{N-n_{23}} X_2 + S^{N-n_{33}} X_3 + S^{N-n_{43}} X_4$$

$$Y_4 = S^{N-n_{34}} X_3 + S^{N-n_{44}} X_4$$

$$Y_1 = S^{N-n_{11}} X_1 + S^{N-n_{21}} X_2 + S^{N-n_{41}} X_4$$

$$Y_2 = S^{N-n_{12}} X_1 + S^{N-n_{22}} X_2 + S^{N-n_{32}} X_3$$

$$Y_3 = S^{N-n_{23}} X_2 + S^{N-n_{33}} X_3 + S^{N-n_{43}} X_4$$

$$Y_4 = S^{N-n_{14}} X_1 + S^{N-n_{34}} X_3 + S^{N-n_{44}} X_4$$

Fig. 2. Three multi-user two-way channel models and two of the classes under consideration.

the channel model (all subsets of  $\{1, 2, 3, 4\}$ ) and  $R_{jk}$  is the rate of transmission from node  $j$  to node  $k$ . For example, in the two-way MAC/BC  $R_{12}$  is the rate of message  $M_{12}$  but  $R_{13}$  does not exist as  $M_{13}$  does not exist.

All channels are assumed to be memoryless and at each channel use, described by the input/output relationships in Fig. 2. Let  $X_j$  and  $Y_k$  denote the channel input of node  $j$  and output at node  $k$  used to describe the model (per channel use). Let  $X_{j,i}$  ( $Y_{j,i}$ ) denote the channel input (output) at node  $j$  at channel use  $i$ , and  $X_j^n := (X_{1,1}, X_{1,2}, \dots, X_{1,n})$ . For the binary modulo 2 adder channels the input and output alphabets are  $\{0, 1\}$ , and  $\oplus$  denotes modulo 2 addition. For the linear deterministic models, the channel inputs and outputs are binary vectors, and all addition will be bit-wise and modulo 2. We furthermore let  $S$  denote an  $N \times N$  lower shift matrix, where  $N = \max(n_{jk})$  over the relevant  $j, k$  for each channel model, where  $n_{jk}$  defines the number of signal bit levels from transmitter  $j$  to receiver  $k$ . We will also consider two other types of channel models: the “deterministic, invertible and cardinality constrained” deterministic channel models and the Gaussian two-way interference channel. We will define those channel models in the appropriate sections.

A node  $j$  is said to employ *adaptation* or *interaction* if the channel input at time  $i$  is a function of the previously received outputs,

$$X_{j,i} = f_j(M_{jk}, Y_j^{i-1}), \quad (1)$$

where  $f_j$  ( $j, k \in \{1, 2, 3, 4\}$  according to the channel model) are deterministic functions. If a node behaves



in a *non-adaptive* or *restricted* fashion then its inputs are functions of its messages only, i.e.  $X_{j,i} = f_j(M_{jk})$ . If some nodes adapt while the others do not, we refer to this as *partial adaptation*, and will specify which nodes adapt. Thus, unless otherwise noted (in the linear deterministic and Gaussian interference channel where there is partial adaptation in some cases), at each time step  $0 \leq i \leq n$ , for  $n$  the blocklength, encoding functions are specified by the mappings  $X_{j,i}(M_{jk}, Y_j^{i-1})$  (for the appropriate message(s)  $M_{j,k}$  known at node  $j$  – 1 message for the MAC/BC and IC channels, 2 messages for the Z channel). Receiver  $k$  uses a decoding function  $g_k : \mathcal{Y}_k^n \times \mathcal{M}_{ki} \rightarrow \widehat{\mathcal{M}}_{jk}$  to obtain an estimate  $\widehat{M}_{jk}$  of the transmitted message  $M_{jk}$  given knowledge of its own message(s)  $M_{ki}$  for suitable  $i$  (depends on model). The capacity region of each channel model is the supremum over all rate tuples for which there exist encoding and decoding functions (of the appropriate rates) which simultaneously drive the probability that any of the estimated messages is not equal to the true message, to zero as  $n \rightarrow \infty$ .

### III. TWO-WAY MAC/BC

We first consider a 3 user, full-duplex two-way network as shown in Fig. 2(a) which forms a multiple access channel in one direction, and a broadcast channel in the other: a Multiple Access/Broadcast Channel (MAC/BC). As introductory example, we first show that adaptation is useless for the modulo 2 adder MAC/BC. We then proceed to a slight generalization of the modulo 2 adder channel: a class of deterministic channels for which we may show that adaptation is useless and capacity is achieved via time-sharing as in the modulo 2 adder channel. Finally, we consider the linear deterministic two-way MAC/BC and show that, once again, adaptation is useless.

#### A. An Introductory Example: Modulo 2 Adder MAC/BC

In the two-way modulo 2 adder MAC/BC, we emphasize that we permit all three users to employ full adaptation – i.e. all channel inputs at time  $i$  may be a function of previously received channel outputs at that node. There are no additional orthogonal, or free, “feedback” links. The capacity region may be stated as follows.

*Theorem 1:* The capacity region of the two-way modulo 2 adder MAC/BC channel is the set of non-negative rate tuples  $(R_{12}, R_{32}, R_{21}, R_{23})$  such that

$$R_{12} + R_{32} \leq 1 \quad (2)$$

$$R_{23} + R_{21} \leq 1. \quad (3)$$

The proof may be found in Appendix VIII-A. The converse may follow by the cut-set bound, but we include an alternative derivation in which we see the role of adaptation and how it may be incorporated into the converse; similar steps are made in the more general models considered later on.

*Remark 1:* The capacity region of Theorem 1 is the same as that of a modulo 2 adder MAC and a modulo 2 adder BC channel in parallel, which do *not* interact. That is, the capacity of a one-way modulo 2 adder MAC is  $R_{12} + R_{32} \leq 1$ , while that of a one-way modulo 2 adder BC (which is actually just a BC with  $Y_1 = X_2 = Y_3$  is  $R_{23} + R_{21} \leq 1$ . No adaptation is needed to achieve these regions. In fact, we notice that capacity is achieved by time-sharing amongst the data traveling in the same “direction” (i.e. between nodes 1 and 3, and between messages  $M_{21}$  and  $M_{23}$ ) but not between the two directions themselves. That is, transmission may take place simultaneously between the two directions, as is the case in the point-to-point modulo 2 adder and Gaussian channel models, where no time-sharing is needed between the two transmission directions  $\rightarrow$  and  $\leftarrow$ .

### B. A more general model for deterministic MAC/BC

Adaptation is useless for the simple modulo 2 adder MAC/BC channel and capacity is achieved using time-sharing in each direction. We ask whether there exists a larger class of channels for which this holds. We answer this positively by considering a class of deterministic two-way MAC/BC channels (which we term “deterministic, invertible and alphabet restricted”) with:

$$\begin{aligned} Y_1 &= F_1(X_1, X_2) \\ Y_2 &= F_2(X_1, X_2, X_3) \\ Y_3 &= F_3(X_2, X_3) \end{aligned}$$

where  $F_m(), m \in \{1, 2, 3\}$  are deterministic functions which also satisfy

- **P1:**  $|\mathcal{X}_1| = |\mathcal{X}_2| = |\mathcal{X}_3| = |\mathcal{X}| = |\mathcal{Y}_1| = |\mathcal{Y}_2| = |\mathcal{Y}_3| = |\mathcal{Y}| = \kappa$  for known  $\kappa \in \mathbb{N}^+$ .
- **P2:** Given  $X_1$ ,  $Y_1$  is invertible, i.e.  $\exists$  a function  $G_1$  s.t.  $X_2 = G_1(X_1, Y_1)$ . Similarly, we assume  $\exists G_{21}, G_{23}, G_3$ :  $X_1 = G_{21}(X_2, X_3, Y_2)$ ,  $X_3 = G_{23}(X_1, X_3, Y_2)$ , and  $X_2 = G_3(X_3, Y_3)$ . This condition excludes two-way channels such as the binary multiplier channel.
- **P3:**  $\exists x_3^*$  such that given  $X_3 = x_3^*$ ,  $X_1, X_2$  both uniform on their alphabets implies both  $Y_1$  and  $Y_2$  uniform on their alphabets. Similarly,  $\exists x_1^*$  such that given  $X_1 = x_1^*$ ,  $X_3, X_2$  both uniform on their alphabets implies  $Y_2, Y_3$  uniform on their alphabets. This ensures we can achieve the full  $\log(\kappa)$ , and is true only for channels with a high degree of symmetry.

Under these conditions (which we only claim are sufficient and not necessarily necessary), the capacity region of the deterministic MAC/BC is given in the following Theorem.

*Theorem 2:* The capacity region of the two-way “deterministic, invertible and alphabet restricted”

MAC/BC satisfying the conditions **P1**, **P2**, **P3** is the set of non-negative  $(R_{12}, R_{32}, R_{21}, R_{43})$  satisfying:

$$R_{12} + R_{32} \leq \log \kappa \quad (4)$$

$$R_{21} + R_{23} \leq \log \kappa, \quad (5)$$

which may be achieved via time-sharing (in each direction).

The proof may be found in Appendix VIII-B. The outer bound is similar to the that of the modulo 2 adder MAC/BC and also follows from the cut-set. Achievability is guaranteed by the “deterministic, invertible and alphabet restricted” conditions. The restriction on the alphabet sizes condition **P1** prohibits “coherent gain” - like phenomena in the outer bound, where correlation between user inputs may be beneficial, as in for example the Gaussian MAC channel with feedback. The remaining two conditions (admittedly quite stringent and symmetric) ensure that we may achieve the outer bound by time-sharing the inputs and selecting the inputs to be uniform.

One example, besides the binary modulo 2 adder channel, is the channel with input and alphabets  $\{0, 1, \dots, \kappa - 1\}$  for some  $\kappa$  and  $Y_1 = X_1 + X_2 \mod \kappa$ ,  $Y_2 = X_1 + X_2 + X_3 \mod \kappa$ , and  $Y_3 = X_2 + X_3 \mod \kappa$ ; we see that adaptation is useless and that the capacity region may be achieved by time-sharing in the forward direction, in parallel with time-sharing in the backwards direction.

*Remark 2:* We comment on restricting the cardinality of the input (which we stress, may not be necessary, but all we seek is another example of a class of channels, for which adaptation is useless). This restriction was brought about by simply considering the two-way MAC/BC binary adder channel (not modulo), with inputs  $\mathcal{X}_1 = \mathcal{X}_3 = \{0, 1\}$  and outputs  $Y_2 = X_1 + X_3$  with alphabet  $\{0, 1, 2\}$  in the MAC direction. In this channel model, it is easy to derive inner and outer bounds both of the form  $R_{12} + R_{32} \leq H(X_1 + X_3)$ . In general, one would hope, like Shannon did for the point-to-point two-way channel [3], to derive multi-user inner and outer bounds of the same *form*. However, even if one is able to do so (which may be too much to hope for in general, but may be reasonable for certain classes of deterministic models), we are left with the distributions over which these bounds are taken. That is, back to our example, to claim that adaptation is useless, the inner bound should be taken over independent input distributions  $p(x_1)p(x_3)$ , while the outer bound, in general allowing for adaptation, is taken over joint distributions  $p(x_1, x_3)$  (unless one restricts the set of input distributions perhaps via dependence-balance-bound-like techniques [12], an open problem). Inner and outer bounds taken over these different sets of distributions do *not* match for the binary adder channel with ternary output. As such, we seek to restrict the channel to those for which a form of cooperation (or adaptation) between users cannot possibly help – which is the case for the modulo 2 adder channels, and as we will see, the similar, in terms of properties, linear deterministic channels.

### C. Linear Deterministic MAC/BC

The two-way linear deterministic MAC/BC channel is defined by the input/output equations as in Fig. 2(a). We recall that all nodes are permitted to adapt, so that at channel use  $i$ ,  $X_{1,i} = f_1(M_{12}, Y_1^{i-1})$ ,  $X_{2,i} = f_2(M_{21}, M_{23}, Y_2^{i-1})$ , and  $X_{3,i} = f_3(M_{32}, Y_3^{i-1})$ . The capacity region may be stated as follows:

*Theorem 3:* The capacity region of the two-way linear deterministic MAC/BC is the set of non-negative rate tuples  $(R_{12}, R_{32}, R_{21}, R_{23})$  such that

$$\text{MAC} \rightarrow \begin{cases} R_{12} \leq n_{12}, & R_{32} \leq n_{32}, \\ R_{12} + R_{32} \leq \max(n_{12}, n_{32}) \end{cases} \quad (6)$$

$$\text{BC} \leftarrow \begin{cases} R_{21} \leq n_{21}, & R_{23} \leq n_{23} \\ R_{21} + R_{23} \leq \max(n_{21}, n_{23}). \end{cases} \quad (7)$$

*Proof:* Achievability may be argued via [6] by mimicking a one-way MAC and one-way BC channel in opposite directions and noting that in this channel model, each user may subtract off its own transmitted signal from its received signal. We note that for this channel, the bounds may be obtained by the cut-set outer bound, but that we derive it in an alternative way nonetheless as the technique illustrates how adaptation may be taken into account directly.

$$\begin{aligned} n(R_{12} - \epsilon) &\leq I(M_{12}; Y_2^n | M_{21}, M_{23}, M_{32}) \\ &\stackrel{(a)}{\leq} \sum_{i=1}^n [H(Y_{2,i} | Y_2^{i-1}, M_{21}, M_{23}, M_{32}, X_2^i)] \\ &\stackrel{(b)}{\leq} \sum_{i=1}^n [H(Y_{2,i} | Y_2^{i-1}, M_{21}, M_{23}, M_{32}, X_2^i, X_3^i)] \\ &\leq \sum_{i=1}^n [H(S^{N-n_{12}} X_{1,i})] \leq n(n_{12}), \end{aligned}$$

where (a) follows since given  $(Y_2^{i-1}, M_{21}, M_{23})$ , we may construct  $X_2^i$ , which cancels out the “self-interference” term  $X_{2,i}$  in  $Y_{2,i}$ . We note that the self-interference term can be always cancelled out in this way in the converse of additive models. Step (b) follows from the fact that given  $M_{32}, X_2^i$ , we may construct  $X_3^i$ . The other single rate bounds follow similarly.

$$\begin{aligned} n(R_{12} + R_{32} - \epsilon) &\leq I(M_{12}, M_{32}; Y_2^n | M_{21}, M_{23}) \\ &\leq \sum_{i=1}^n [H(Y_{2,i} | Y_2^{i-1}, M_{21}, M_{23}, X_2^i)] \\ &\leq \sum_{i=1}^n [H(S^{N-n_{12}} X_{1,i} + S^{N-n_{32}} X_{3,i})] \leq n(\max(n_{12}, n_{32})). \end{aligned}$$

We may analogously obtain the other sum-rate bound. ■

*Remark 3:* Without adaptation, the channel would correspond to a MAC channel simultaneously transmitting with a BC channel with restricted nodes. Since, even with adaptation, we are able to achieve the desired rates in one channel use, adaptation is useless, and the capacity region is a four dimensional region that is equivalent to the capacity region of the linear deterministic MAC and the linear deterministic BC in opposite directions.

#### IV. TWO-WAY Z CHANNEL

We now consider the 4 user, full-duplex network as shown in Fig. 2(b). The 6 message network, resembles a cascade of three two-way channels, in the shape of a Z (in each direction). Again, we first introduce the modulo 2 adder model, show that adaptation is useless, mention a generalization to a slightly larger class of deterministic channels, and then show the same is true for the linear deterministic two-way Z channel.

##### A. An Introductory Example: Modulo 2 Adder Two-way Z Channel

The two-way modulo 2 adder Z channel is discrete and memoryless, and all four users may employ full adaptation. The capacity region of this channel is stated as follows:

*Theorem 4:* The capacity region of the two-way modulo 2 adder Z channel is the set of non-negative rate tuples  $(R_{12}, R_{21}, R_{23}, R_{32}, R_{34}, R_{43})$  such that

$$R_{12} + R_{32} + R_{34} \leq 1 \quad (8)$$

$$R_{21} + R_{23} + R_{43} \leq 1 \quad (9)$$

The proof is found in Appendix VIII-C and is not a direct consequence of the cut-set outer bound.

*Remark 4:* We again notice that since time-sharing achieves the above region, adaptation does not enlarge the capacity region. We again see that the messages in the  $\rightarrow$  and the  $\leftarrow$  directions may be simultaneously communicated, but that the messages within one direction must be time-shared.

##### B. A More General Model for Two-way Z Channel

Similar to the more general “deterministic, invertible and restricted” class of two-way MAC/BC channels where it was shown that non-adaptive time-sharing achieves capacity, we may extend the two-way Z modulo 2 adder model to a more general class of two-way Z channels. The converse follows along similar lines as for the modulo 2 adder channel. In terms of achieving the outer bounds  $R_{12} + R_{32} + R_{34} \leq \log \kappa$  and  $R_{21} + R_{23} + R_{43} \leq \log \kappa$ , one sufficient condition involves restricting the input and output alphabet sizes to be equal (eliminating some of the potential benefits of adaptation via user cooperation), as well as several symmetry constraints akin to extensions of **P2** and **P3**. Again, one example of such a channel

model is the modulo  $\kappa$  channel. We omit the full statement as it follows in a straightforward and analogous fashion to Theorems 2 and 4.

### C. Linear Deterministic Two-way Z Channel

The linear deterministic two-way Z channel is defined by the input / output equations in Fig. 2(b), where we recall that all nodes may employ adaptation. In this case, the capacity region is again that of two parallel Z channels in opposite directions; adaptation is useless.

*Theorem 5:* The capacity region of the two-way linear deterministic Z channel is the set of all rate-tuples  $(R_{12}, R_{21}, R_{23}, R_{32}, R_{34}, R_{43})$  which satisfy the following:

$$\begin{aligned} \mathbf{Z} &\rightarrow \begin{cases} R_{12} \leq n_{12}, & R_{32} \leq n_{32}, & R_{34} \leq n_{34} \\ R_{12} + R_{32} \leq \max(n_{12}, n_{32}) \\ R_{32} + R_{34} \leq \max(n_{32}, n_{34}) \\ R_{12} + R_{32} + R_{34} \leq \max(n_{12}, n_{32}) + [n_{34} - n_{32}]^+ \end{cases} \\ \mathbf{Z} &\leftarrow \begin{cases} R_{43} \leq n_{43}, & R_{23} \leq n_{23}, & R_{21} \leq n_{21} \\ R_{43} + R_{23} \leq \max(n_{43}, n_{23}) \\ R_{23} + R_{21} \leq \max(n_{23}, n_{21}) \\ R_{43} + R_{23} + R_{21} \leq \max(n_{43}, n_{23}) + [n_{21} - n_{23}]^+. \end{cases} \end{aligned}$$

*Proof:* We first note that the capacity of a class of deterministic Z channels is shown in [19, Th. 3.1]. To show achievability of the above, we use the achievability scheme of [19, Th. 3.1] in each  $\rightarrow$  and  $\leftarrow$  direction with non-adaptive nodes (adaptive may mimic non-adaptive). Note that due to the additive nature of the channel, each receiver may cancel or subtract out its own “self-interference” term  $S^{N-n_{jj}}X_j$  from its received signal. By making the appropriate correspondences, we see that the above is achievable and equivalent to two one-way Z channels.

Now, we prove the converse. We note that again, all but the triple-rate bounds may be obtained by the two-way cut-set bound, or independently by giving the appropriate side-information or genie to the receivers (as illustrated in previous models). We omit these out for sake of repetition. The non-cut-set triple rate bound may be obtained as follows:

$$\begin{aligned} n(R_{12} + R_{32} + R_{34} - \epsilon) &\leq I(M_{12}; Y_2^n | M_{21}, M_{23}, M_{43}) + I(M_{32}, M_{34}; Y_2^n, Y_4^n | M_{43}, M_{12}, M_{21}, M_{23}) \\ &\leq H(Y_2^n | M_{21}, M_{23}, M_{43}) + H(Y_4^n | M_{43}, M_{12}, M_{21}, M_{23}, Y_2^n) \\ &\leq \sum_{i=1}^n [H(Y_{2,i} | Y_2^{i-1}, M_{21}, M_{23}, X_2^i) + H(Y_{4,i} | M_{12}, M_{21}, M_{23}, M_{43}, Y_4^{i-1}, X_4^i, Y_2^n, X_2^n)] \end{aligned}$$

$$\begin{aligned}
& \stackrel{(a)}{\leq} \sum_{i=1}^n [H(S^{N-n_{12}}X_{1,i} + S^{N-n_{32}}X_{3,i}) \\
& \quad + H(S^{N-n_{34}}X_{3,i} | M_{12}, M_{21}, M_{23}, M_{43}, Y_4^{i-1}, X_4^i, S^{N-n_{12}}X_{1,i} + S^{N-n_{22}}X_{2,i} + S^{N-n_{32}}X_{3,i}, X_2^n, X_1^n)] \\
& \leq \sum_{i=1}^n [H(S^{N-n_{12}}X_{1,i} + S^{N-n_{32}}X_{3,i}) + H(S^{N-n_{34}}X_{3,i} | S^{N-n_{32}}X_{3,i})] \\
& \leq n(\max(n_{12}, n_{32}) + [n_{34} - n_{32}]^+).
\end{aligned}$$

In (a),  $X_1^n$  in the second entropy term follows since given,  $M_{12}$  and  $X_2^n$ , we may construct  $X_1^n$ . ■

*Remark 5:* Again, we are always able to achieve the desired rates in Theorem 5 in only one channel use, therefore adaptation is useless. The capacity region of this channel, a 6 dimensional region, is exactly equivalent to the capacity region of the two one-way linear deterministic Z channels.

## V. DETERMINISTIC TWO-WAY INTERFERENCE CHANNELS

The last deterministic multi-user two-way network we consider is a 4 user, 4 message, full-duplex network as shown in Fig. 2(c). This channel model merges elements of two-way, feedback, and interference, and forms two parallel interference channels in the  $\rightarrow$  and  $\leftarrow$  directions. Again, we first introduce the modulo 2 adder model of this channel and show that adaptation is useless, generalizing this to a slightly larger class of symmetric channels. This generalization is not as straightforward as for the MAC/BC and Z channels, and hence is discussed in somewhat more depth. Finally, for the symmetric linear deterministic two-way interference channel, we show that full adaptation is useless when the interference is very strong, strong, and in some of the weak regimes, while in all other regimes we show that partial adaptation is useless (i.e. if only 2 of the nodes adapt, might as well have none of the nodes adapt).

### A. An Introductory Example: Modulo 2 Adder Two-way IC

Similar to the MAC/BC and Z channels, we are motivated by the two-way, modulo 2 adder IC, perhaps the simplest example of a two-way IC channel in which adaptation is useless, and capacity is achieved through time-sharing.

*Theorem 6:* The capacity region of the two-way modulo 2 adder interference channel is the set of non-negative rate tuples  $(R_{12}, R_{21}, R_{34}, R_{43})$  such that

$$R_{12} + R_{34} \leq 1 \tag{10}$$

$$R_{21} + R_{43} \leq 1. \tag{11}$$

The proof is provided in Appendix VIII-D.

*Remark 6:* The capacity region of the two-way modulo 2 adder interference channel with full adaptation is the same as the combination of two one-way modulo 2 adder interference channels. Capacity is achieved using time sharing and nodes need not adapt. Thus, adaptation is again useless in this scenario.

### B. Comments on a more general class of two-way deterministic ICs

We ask whether the above two-way modulo 2 IC results may be extended to a more general class of deterministic ICs in which adaptation is useless and capacity is achieved through time-sharing. In both the MAC/BC and Z channel models we were able to accomplish this by imposing certain cardinality, invertibility and symmetry constraints. One example of a channel in this class is the modulo- $\kappa$  (for some  $\kappa$ ) channel. We now extend results to the two-way IC, but note that we must make two additional restrictions: 1) we do not consider “self-interference” (which we did in the previous two models), and 2) we impose symmetry of the outputs (common output in each direction). Both of these conditions are sufficient for obtaining sum-rate outer bounds equal to  $\log \kappa$  in each direction (where  $\kappa$  is the input/output alphabet size); whether they are necessary remains open.

Consider a class of deterministic two-way interference channels without self-interference, described by:

$$Y_1 = F_{\rightarrow}(X_2, X_4) = Y_3 \text{ (there is no self-interference, symmetric channel)}$$

$$Y_2 = F_{\leftarrow}(X_1, X_3) = Y_4 \text{ (there is no self-interference, symmetric channel)}$$

where  $F_{\rightarrow}, F_{\leftarrow}$  are deterministic functions. Further restrict the class of channels to those with:

- **P1IC:**  $|\mathcal{X}_1| = |\mathcal{X}_2| = |\mathcal{X}_3| = |\mathcal{X}_4| = |\mathcal{Y}_1| = |\mathcal{Y}_2| = |\mathcal{Y}_3| = |\mathcal{Y}_4| = \kappa$  for known  $\kappa \in \mathbb{N}^+$ .
- **P2IC:** “Invertibility” constraints reminiscent of Costa and El Gamal [18]. In the notation of [18], we assume  $f_1 = f_2 = F_{\rightarrow}$  (and similarly, in the reverse direction we have  $f_1 = f_2 = F_{\leftarrow}$ ), and that  $g_1 = g_2$  are the identity functions, i.e.  $g_1(X_1) = X_1$  and  $g_2(X_3) = X_3$  (and similarly for the reverse direction). Then we require that, given  $X_1, Y_2$  is invertible, i.e.  $\exists$  a function  $G_2$  s.t.  $X_3 = G_2(X_1, Y_2)$ . Similarly, we assume  $\exists G_1, G_3, G_4$ :  $X_4 = G_1(X_2, Y_1)$ ,  $X_2 = G_3(X_4, Y_3)$ , and  $X_1 = G_4(X_3, Y_4)$ .
- **P3IC:** to ensure that we may attain the outer bound through time-sharing, we impose that  $F_{\rightarrow}$  is a function such that  $\exists x_3^*$  such that  $X_1$  and  $Y_2 = Y_4 = F_{\rightarrow}(X_1, X_3 = x_3^*)$  are in 1-to-1 correspondence, and  $\exists x_1^*$  such that  $X_3$  and  $Y_2 = Y_4 = F_{\rightarrow}(X_1 = x_1^*, X_3)$  are in 1-to-1 correspondence, (and similarly for  $F_{\leftarrow}$ ).

For this class of channels, the capacity is given by the following:

*Theorem 7:* The capacity region of the two-way “deterministic, invertible and alphabet restricted” IC



satisfying the conditions **P1IC**, **P2IC**, **P3IC** is the set of non-negative rates  $(R_{12}, R_{34}, R_{21}, R_{43})$  satisfying:

$$R_{12} + R_{34} \leq \log \kappa \quad (12)$$

$$R_{21} + R_{43} \leq \log \kappa, \quad (13)$$

which may be achieved via time-sharing (in each direction).

*Proof:* Let us consider only the  $\rightarrow$  direction for now. Under the above restrictions, the capacity region of the class of deterministic (one-way) interference channels in [18] may be simplified to

$$R_{12} + R_{34} \leq \log \kappa \quad (14)$$

which may be achieved by time-sharing between the inputs  $X_1$  uniform over the  $\kappa$  input symbols, while  $X_3 = x_3^*$  and vice versa. That the rates (14) are achievable may alternatively be directly verified.

We find the matching outer bound:

$$\begin{aligned} & n(R_{12} + R_{34} - \epsilon) \\ & \leq I(M_{12}; Y_2^n | M_{21}, M_{43}) + I(M_{34}; Y_4^n, Y_2^n | M_{12}, M_{21}, M_{43}) \\ & = I(M_{12}; Y_2^n | M_{21}, M_{43}) + I(M_{34}; Y_2^n | M_{21}, M_{12}, M_{43}) + I(M_{34}; Y_4^n | M_{21}, M_{12}, M_{43}, Y_2^n) \\ & \stackrel{(a)}{\leq} \sum_{i=1}^n [H(Y_{2,i} | Y_2^{i-1}, M_{21}, M_{43}) - H(Y_{2,i} | Y_2^{i-1}, M_{12}, M_{21}, M_{43}) + H(Y_{2,i} | Y_2^{i-1}, M_{12}, M_{21}, M_{43})] \\ & \leq \sum_{i=1}^n [H(Y_{2,i})] \\ & = n \max_{p(x_1, x_3)} H(Y_2) \leq n \log \kappa, \end{aligned}$$

where in (a) we dropped two negative entropy terms, and were able to replace  $Y_{4,i}$  by  $Y_{2,i}$ , allowing us to cancel the 2nd and 3rd terms. This is the central reason why we have restricted  $Y_2 = Y_4$  and  $Y_1 = Y_3$ , whether one may somehow cancel these terms when this is not the case is open. Restricting the alphabet size as in **P1IC** yields the final inequality. ■

*Remark 7:* We have proposed a slightly more general model for deterministic two-way ICs in which adaptation is useless. However, it should be pointed out that our conditions are sufficient but by no means necessary. For instance, consider a binary multiplier two-way interference channel described by  $Y_1 = Y_3 = X_2 X_4$  and  $Y_2 = Y_4 = X_1 X_3$ , with all inputs and outputs binary. It is not difficult to show that adaptation is useless for this model and the capacity of this channel is equivalent to the capacity of two one-way binary multiplier interference channels in parallel, the same capacity region as in Theorem 6. In addition, we will show in Section VI that adaptation is also useless for the Gaussian two-way interference channel with partial adaptation when the two-way interference is very strong; this channel is not in the class of channels considered above either.

### C. Linear Deterministic Two-way IC

The two-way linear deterministic interference channel is defined by the input / output equations in Fig. 2(c). In this section we will be considering the general linear deterministic IC, as well as the “symmetric” linear deterministic IC for which  $p := n_{12} = n_{21} = n_{34} = n_{43}$ ,  $q := n_{14} = n_{41} = n_{23} = n_{32}$ , and  $\alpha := q/p$ . This will allow us to compare the symmetric, normalized sum capacity of various one and two-way interference channels, defined as  $C_{sym}(\alpha) := \frac{R_{12}+R_{34}}{2}$ .

Recall our definition of partial adaptation (nodes 1 and 3 are fixed or “restricted”) of Section II:

$$X_{1,i} = f_1(M_{12}), \quad X_{2,i} = f_2(M_{21}, Y_2^{i-1}) \quad (15)$$

$$X_{3,i} = f_3(M_{34}), \quad X_{4,i} = f_4(M_{43}, Y_4^{i-1}) \quad (16)$$

We first prove a Lemma regarding partial adaptation, which is key in showing that partial adaptation is useless, and that the inability of *certain* nodes to adapt essentially “blocks” the ability of adaptation to help at all.

*Lemma 8:* Under partial adaptation conditions (15) – (16), for some deterministic functions  $f_5$  and  $f_6$ ,

$$X_{2,i} = f_5(M_{12}, M_{21}, M_{34}) \perp M_{43}, \quad \forall i \quad (17)$$

$$X_{4,i} = f_6(M_{43}, M_{34}, M_{12}) \perp M_{21}, \quad \forall i \quad (18)$$

where  $\perp$  denotes independence.

*Proof:* Note that  $X_{2,i} = f_2(M_{21}, Y_2^{i-1})$  and  $Y_2^{i-1} = S^{N-n_{12}} X_1^{i-1} + S^{N-n_{22}} X_2^{i-1} + S^{N-n_{32}} X_3^{i-1}$ . Since  $X_1^{i-1}$  and  $X_3^{i-1}$  are functions only of  $M_{12}$  and  $M_{34}$  respectively, we may conclude that there exists a function  $f^*$  such that  $X_{2,i} = f^*(M_{21}, M_{12}, M_{34}, X_2^{i-1})$ . Iterating this argument, and noting that  $X_{2,1}$  is only a function of  $M_{21}$ , we obtain the theorem. The result for  $X_{4,i}$  follows by a similar argument. ■

*Theorem 9:* The capacity region of the two-way linear deterministic interference channel under partial adaptation constraints is the set of  $(R_{12}, R_{21}, R_{34}, R_{43})$  which satisfy the following:

$$R_{12} \leq n_{12}, \quad R_{34} \leq n_{34} \quad (\text{IC} \rightarrow \text{a})$$

$$R_{12} + R_{34} \leq \max(n_{12}, n_{32}) + [n_{34} - n_{32}]^+ \quad (\text{IC} \rightarrow \text{b})$$

$$R_{12} + R_{34} \leq \max(n_{34}, n_{14}) + [n_{12} - n_{14}]^+ \quad (\text{IC} \rightarrow \text{c})$$

$$R_{12} + R_{34} \leq \max([n_{12} - n_{14}]^+, n_{32}) + \max([n_{34} - n_{32}]^+, n_{14}) \quad (\text{IC} \rightarrow \text{d})$$

$$2R_{12} + R_{34} \leq \max(n_{12}, n_{32}) + [n_{12} - n_{14}]^+ + \max([n_{34} - n_{32}]^+, n_{14}) \quad (\text{IC} \rightarrow \text{e})$$

$$R_{12} + 2R_{34} \leq \max(n_{34}, n_{14}) + [n_{34} - n_{32}]^+ + \max([n_{12} - n_{14}]^+, n_{32}) \quad (\text{IC} \rightarrow \text{f})$$

$$R_{21} \leq n_{21}, \quad R_{43} \leq n_{43} \quad (\text{IC} \leftarrow \text{a})$$

$$R_{21} + R_{43} \leq \max(n_{21}, n_{41}) + [n_{43} - n_{41}]^+ \quad (\text{IC} \leftarrow \text{b})$$

$$R_{21} + R_{43} \leq \max(n_{43}, n_{23}) + [n_{21} - n_{23}]^+ \quad (\text{IC} \leftarrow \text{c})$$

$$R_{21} + R_{43} \leq \max([n_{21} - n_{23}]^+, n_{41}) + \max([n_{43} - n_{41}]^+, n_{23}) \quad (\text{IC} \leftarrow \text{d})$$

$$2R_{21} + R_{43} \leq \max(n_{21}, n_{41}) + [n_{21} - n_{23}]^+ + \max([n_{43} - n_{41}]^+, n_{23}) \quad (\text{IC} \leftarrow \text{e})$$

$$R_{21} + 2R_{43} \leq \max(n_{43}, n_{23}) + [n_{43} - n_{41}]^+ + \max([n_{21} - n_{23}]^+, n_{41}). \quad (\text{IC} \leftarrow \text{f})$$

*Proof:* For achievability, note that self-interference may be removed at each receiver due to this channel model's linearity, in which case the physical channel model reduces to two one-way IC in opposite directions. We may thus apply the well-known Han-Kobayashi scheme [27] in each direction, ignoring the ability of the nodes to adapt, achieving the expression in (IC $\rightarrow$ ) and (IC $\leftarrow$ ).

Now we prove the converse. Single-rates follow as in (6), and using Lemma 8 (where we use partial adaptation). For the sum-rates (IC $\rightarrow$  b):

$$\begin{aligned} & n(R_{12} + R_{34} - \epsilon) \\ & \stackrel{(a)}{\leq} I(M_{12}; Y_2^n | M_{21}, M_{43}) + I(M_{34}; Y_4^n, Y_2^n | M_{12}, M_{21}, M_{43}) \\ & \leq I(M_{12}; Y_2^n | M_{21}, M_{43}) + I(M_{34}; Y_2^n | M_{21}, M_{12}, M_{43}) + H(Y_4^n | M_{21}, M_{12}, M_{43}, Y_2^n) \\ & \stackrel{(b)}{=} I(M_{12}; Y_2^n | M_{21}, M_{43}) + I(M_{34}; Y_2^n | M_{21}, M_{12}, M_{43}) \\ & + \sum_{i=1}^n [H(S^{N-n_{34}} X_{3,i} | M_{21}, M_{12}, M_{43}, Y_4^{i-1}, X_4^i, Y_2^n, X_2^n, X_1^i)] \\ & \leq \sum_{i=1}^n [H(Y_{2,i} | Y_2^{i-1}, M_{21}, X_2^i) - H(Y_{2,i} | Y_2^{i-1}, M_{12}, M_{21}, M_{43}) + H(Y_{2,i} | Y_2^{i-1}, M_{12}, M_{21}, M_{43}) \\ & + H(S^{N-n_{34}} X_{3,i} | M_{21}, M_{12}, M_{43}, Y_4^{i-1}, X_4^i, S^{N-n_{12}} X_1^n + S^{N-n_{22}} X_2^n + S^{N-n_{32}} X_3^n, X_2^n, X_1^i)] \\ & \leq \sum_{i=1}^n [H(S^{N-n_{12}} X_{1,i} + S^{N-n_{32}} X_{3,i}) + H(S^{N-n_{34}} X_{3,i} | S^{N-n_{32}} X_{3,i})] \\ & \leq n(\max(n_{12}, n_{32}) + [n_{34} - n_{32}]^+) \\ & \stackrel{(c)}{=} n(\max(p, q) + [p - q]^+). \end{aligned}$$

We introduce the genie  $Y_2^n$  in the second mutual information term in (a), i.e. we provide asymmetric side information to only one receiver. In (b), we add  $X_1^i$  in the entropy term because of the iterated argument that, given  $M_{12}, X_2^n, X_4^i$ , we can construct  $X_1^i$ . For (c), we assumed a symmetric channel.

*Remark 8:* Note that we do **not** need partial adaptation in this bound, and so these conclusions actually hold for **full adaptation**. This implies that for the symmetric channel, full adaptation is useless when two-way interference is strong ( $1 \leq \alpha \leq 2, \alpha = q/p$ ) and weak in some interval ( $2/3 \leq \alpha \leq 1, \alpha = q/p$ )

where this outer bound may be achieved. Interestingly, when  $2/3 \leq \alpha \leq 2$ , the “V” curve is also the capacity for the linear deterministic symmetric interference channel with feedback [21]. If we add another asymmetric genie  $Y_4^n$  in the first term in (a), then we obtain the second sum-rate bound (IC  $\rightarrow$  c).

It may further be shown that for symmetric channels, adaptation is also useless when two-way interference is very strong ( $\alpha > 2, \alpha = q/p$ ). To show this, we re-derive the single-rate bounds this time *not* assuming partial adaptation (allowing for full adaptation), and using symmetry in the last step:

$$\begin{aligned}
n(R_{12} - \epsilon) &\leq I(M_{12}; Y_2^n, Y_3^n | M_{21}, M_{34}) \\
&\leq H(Y_2^n, Y_3^n | M_{21}, M_{34}) \\
&= \sum_{i=1}^n [H(Y_{2,i}, Y_{3,i} | Y_2^{i-1}, Y_3^{i-1}, M_{21}, M_{34}, X_2^i, X_3^i)] \\
&= \sum_{i=1}^n [H(S^{N-n_{12}} X_{1,i}, S^{N-n_{43}} X_{4,i} | Y_2^{i-1}, Y_3^{i-1}, M_{21}, M_{34}, X_2^i, X_3^i)] \\
&\leq \sum_{i=1}^n [H(S^{N-n_{12}} X_{1,i}, S^{N-n_{43}} X_{4,i})] \\
&= n \max(n_{12}, n_{43}) \\
&= np
\end{aligned}$$

Under very strong interference constraints, this is also known to be achievable. Thus, we have obtained the capacity (or generalized degrees of freedom) for the symmetric linear deterministic two-way IC when  $\alpha \geq 2/3$ , where we see that full adaptation is useless. We will comment more on this in Remark V-D, and in Fig. 3.

Continuing on the sum-rate outer bound (IC  $\rightarrow$  d):

$$\begin{aligned}
n(R_{12} + R_{34} - \epsilon) &\leq I(M_{12}; Y_2^n, S^{N-n_{14}} X_1^n, M_{21}, M_{43}) + I(M_{34}; Y_4^n, S^{N-n_{32}} X_3^n, M_{21}, M_{43}) \\
&\stackrel{(d)}{=} H(Y_2^n | S^{N-n_{14}} X_1^n, M_{43}, M_{21}) + H(Y_4^n | S^{N-n_{32}} X_3^n, M_{43}, M_{21}) \\
&\quad + \sum_{i=1}^n H(S^{N-n_{14}} X_{1,i} | S^{N-n_{14}} X_1^{i-1}, M_{43}, M_{21}) - H(Y_{2,i}, S^{N-n_{14}} X_{1,i} | Y_2^{i-1}, S^{N-n_{14}} X_1^{i-1}, M_{12}, M_{21}, M_{43}, X_2^i, X_1^i) \\
&\quad + \sum_{i=1}^n H(S^{N-n_{32}} X_{3,i} | S^{N-n_{32}} X_3^{i-1}, M_{43}, M_{21}) - H(Y_{4,i}, S^{N-n_{32}} X_{3,i} | Y_4^{i-1}, S^{N-n_{32}} X_3^{i-1}, M_{34}, M_{21}, M_{43}, X_4^i, X_3^i) \\
&\stackrel{(e)}{=} H(Y_2^n | S^{N-n_{14}} X_1^n, M_{43}, M_{21}) + H(Y_4^n | S^{N-n_{32}} X_3^n, M_{43}, M_{21}) \\
&\quad + \sum_{i=1}^n H(S^{N-n_{14}} X_{1,i} | S^{N-n_{14}} X_1^{i-1}, M_{43}, M_{21}, M_{34}) - H(S^{N-n_{23}} X_{3,i} | Y_2^{i-1}, S^{N-n_{14}} X_1^{i-1}, M_{12}, M_{21}, M_{43}, X_2^i, X_1^i) \\
&\quad + \sum_{i=1}^n H(S^{N-n_{32}} X_{3,i} | S^{N-n_{32}} X_3^{i-1}, M_{43}, M_{21}, M_{34}) - H(S^{N-n_{14}} X_{1,i} | Y_4^{i-1}, S^{N-n_{32}} X_3^{i-1}, M_{34}, M_{21}, M_{43}, X_4^i, X_3^i)
\end{aligned}$$

$$\begin{aligned}
&= H(Y_2^n | S^{N-n_{14}} X_1^n, M_{43}, M_{21}) + H(Y_4^n | S^{N-n_{32}} X_3^n, M_{43}, M_{21}) \\
&\leq \sum_{i=1}^n [H(S^{N-n_{12}} X_{1,i} + S^{N-n_{32}} X_{3,i} | S^{N-n_{14}} X_{1,i}) + H(S^{N-n_{14}} X_{1,i} + S^{N-n_{34}} X_{4,i} | S^{N-n_{32}} X_{3,i})] \\
&\leq n(\max([n_{12} - n_{14}]^+, n_{32}) + \max([n_{34} - n_{32}]^+, n_{14})),
\end{aligned}$$

where (d) follows from the independence of the messages and the fact that we can create  $X_1^i$  given  $M_{12}$ , and we can create  $X_2^i$  given  $M_{21}, Y_2^{i-1}$  (similarly for  $X_3^i, X_4^i$ ). For (e), the 3rd and 6th terms cancel, as do the 4th and 5th, due to the fact that  $X_{1,i}$  and  $X_{3,i}$  are functions only of their own message, and reducing knowledge of  $Y_4^{i-1}$  to  $S^{N-n_{14}} X_1^{i-1}$  given knowledge of  $X_3^i$  and  $X_4^i$ . Finally,  $Y_4^{i-1}, X_4^i$  can be constructed in the conditioning of the 3rd term (and similarly for the 4th and 5th terms).

The sum-rate bound in the opposite direction (which we must consider given the fact that under partial adaptation, not everything is symmetric):

$$\begin{aligned}
n(R_{21} + R_{43} - \epsilon) &\leq I(M_{21}; Y_1^n, S^{N-n_{23}} X_2^n, M_{12}, M_{34}) + I(M_{43}; Y_3^n, S^{N-n_{41}} X_4^n, M_{12}, M_{34}) \\
&\stackrel{(f)}{=} H(Y_1^n | S^{N-n_{23}} X_2^n, M_{12}, M_{34}) + H(S^{N-n_{23}} X_2^n | M_{12}, M_{34}) - H(Y_1^n, S^{N-n_{23}} X_2^n | M_{12}, M_{34}, M_{21}) \\
&\quad + H(Y_3^n | S^{N-n_{41}} X_4^n, M_{12}, M_{34}) + H(S^{N-n_{41}} X_4^n | M_{12}, M_{34}) - H(Y_3^n, S^{N-n_{41}} X_4^n | M_{12}, M_{34}, M_{43}) \\
&\stackrel{(g)}{=} H(Y_1^n | S^{N-n_{23}} X_2^n, M_{12}, M_{34}) + H(S^{N-n_{23}} X_2^n | M_{12}, M_{34}, M_{43}) - H(S^{N-n_{41}} X_4^n | M_{12}, M_{34}, M_{21}) \\
&\quad + H(Y_3^n | S^{N-n_{41}} X_4^n, M_{12}, M_{34}) + H(S^{N-n_{41}} X_4^n | M_{12}, M_{34}, M_{21}) - H(S^{N-n_{23}} X_2^n | M_{12}, M_{34}, M_{43}) \\
&\leq \sum_{i=1}^n [H(S^{N-n_{21}} X_{2,i} + S^{N-n_{41}} X_{4,i} | S^{N-n_{23}} X_{2,i}) + H(S^{N-n_{43}} X_{4,i} + S^{N-n_{23}} X_{2,i} | S^{N-n_{41}} X_{4,i})] \\
&\leq n(\max([n_{21} - n_{23}]^+, n_{41}) + \max([n_{43} - n_{41}]^+, n_{23})),
\end{aligned}$$

where (f) follows from the independence of the messages. Equation (g) follows since  $X_1$  and  $X_3$  are functions only of  $M_{12}$  and  $M_{34}$  and from Lemma 8.

*Remark 9:* We needed partial adaptation (Lemma 8) in the proof of the previous two bounds (IC $\rightarrow$  d) and (IC $\leftarrow$  d). In the above, nodes 1 and 3 were restricted. By symmetry, we may obtain the same result if nodes 2 and 4 were restricted. Finally,

$$\begin{aligned}
&n(2R_{12} + R_{34} - \epsilon) \\
&\leq I(M_{12}; Y_2^n | M_{21}, M_{43}) + I(M_{12}; Y_2^n, Y_4^n | M_{21}, M_{43}, M_{34}) + I(M_{34}; Y_4^n, S^{N-n_{32}} X_3^n | M_{21}, M_{43}) \\
&= H(Y_2^n | M_{21}, M_{43}) - H(Y_2^n | M_{21}, M_{43}, M_{12}) + H(Y_4^n | M_{21}, M_{43}, M_{34}) \\
&\quad + H(Y_2^n | M_{21}, M_{43}, M_{34}, Y_4^n) + H(Y_4^n, S^{N-n_{32}} X_3^n | M_{21}, M_{43}) - H(Y_4^n, S^{N-n_{32}} X_3^n | M_{34}, M_{21}, M_{43})
\end{aligned}$$

$$\begin{aligned}
&\stackrel{(h)}{=} H(Y_2^n | M_{21}, M_{43}) - H(Y_2^n | M_{21}, M_{43}, M_{12}) + H(S^{N-n_{32}} X_3^n | M_{43}, M_{21}, M_{12}) \\
&+ H(Y_4^n | S^{N-n_{32}} X_3^n, M_{43}, M_{21}) + H(Y_4^n | M_{21}, M_{43}, M_{34}) \\
&- H(Y_4^n, S^{N-n_{32}} X_3^n | M_{34}, M_{21}, M_{43}) + H(Y_2^n | M_{21}, M_{43}, M_{34}, Y_4^n) \\
&\stackrel{(i)}{\leq} \sum_{i=1}^n [H(S^{N-n_{12}} X_{1,i} + S^{N-n_{32}} X_{3,i}) + H(S^{N-n_{14}} X_{1,i} + S^{N-n_{34}} X_{3,i} | S^{N-n_{32}} X_{3,i}) + H(S^{N-n_{12}} X_{1,i} | S^{N-n_{14}} X_{1,i})] \\
&= n(\max(n_{12}, n_{32}) + \max([n_{34} - n_{32}]^+, n_{14}) + [n_{12} - n_{14}]^+),
\end{aligned}$$

where (h) follows from the definition of partial adaptation and Lemma 8 (skipping a transition to multi-letter for brevity), and (i) by canceling the 2nd and 3rd terms, as well as the 5th and the 6th terms. We may similarly prove the other bounds of this form (IC $\rightarrow$  f), (IC $\leftarrow$  e) and (IC $\leftarrow$  f). ■

We again see that, under partial adaptation constraints, adaptation is useless and we obtain the capacity region of two one-way ICs. Essentially, *partial* adaptation prevented messages being relayed by other messages (which was also impossible in the MAC/BC and Z channels). For example, under full adaptation, message  $M_{12}$  may be relayed from Tx1 to Rx 2 through nodes 3 and 4. This path is “blocked” by the partial adaptation assumption, as node 3 could not adapt to carry  $M_{12}$ . However, it should be pointed out that this is not necessary in general: full adaptation in the two-way modulo 2 adder IC is useless as we showed in the previous subsection, but the path is not blocked.

#### D. Symmetric rate comparison with other interference channel models

For symmetric deterministic linear ICs, we may compare the symmetric sum-capacity  $C_{sym}$  of various one-way and two-way models. Recalling  $\alpha := q/p$ , we plot  $C_{sym}$  as a function of  $\alpha$  for the IC [7], IC with noiseless output feedback [21], the IC with rate-limited feedback [22] (for a fixed value of  $\beta = 0.125$  in the notation of [22]), and the two-way IC with full adaptation considered here (for  $\alpha \geq 2/3$  only). Several observations may be made: the two-way IC with *partial* adaptation behaves like two one-way interference channels operating in parallel over the forward and backwards link, as seen by the coinciding lines for the one-way and two-way IC with partial adaptation. This tells us that allowing partial adaptation is useless – i.e. may as well not adapt. Interestingly, the same holds true even for full adaptation for  $\alpha > 2/3$ . This was also concluded for the linear deterministic one-way interference channel with interfering feedback links in [20]; what is interesting is that we can just as well squeeze in extra information messages in the feedback link (in the two-way interference channel model) rather than use the backwards links for feedback. The symmetric sum-capacity for the *fully* adaptive two-way IC remains open for  $\alpha < 2/3$ ; it is solved for partial adaptation.

Recently, the work in [24] has considered a one-way interference channel with interfering feedback links (again forming an interference channel), a generalization of some of the deterministic interference

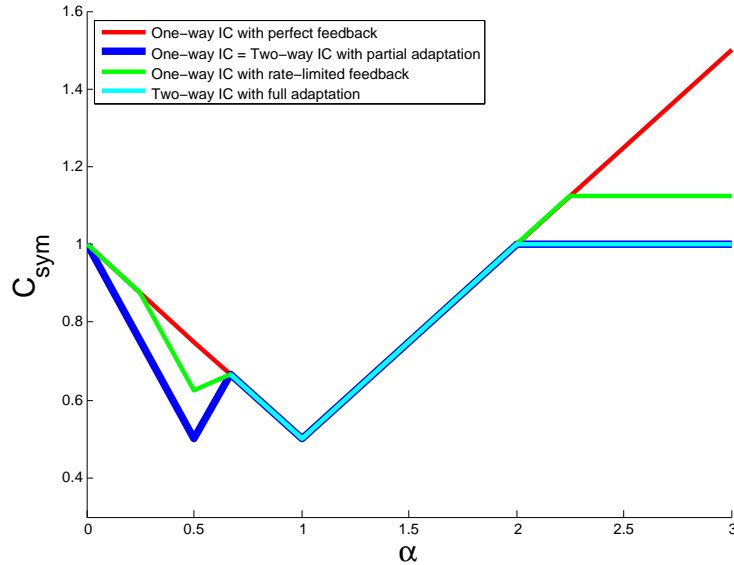


Fig. 3.  $C_{sym}$  for various linear deterministic ICs as a function of  $\alpha := \frac{q}{p}$ ;  $q$  interfering link strength,  $p$  direct link strength.

channels with feedback considered in [20], where the feedback link spends fraction  $\lambda$  of its time sending feedback, and uses the remaining  $(1 - \lambda)$  for other things (such as for example sending independent backwards messages, though adaptation as in (1) is not considered). This is quite different from our model which integrates sending feedback and messages over all links and does not force this separation. While the symmetric sum-capacity for this two-message interference channel with interfering feedback links is obtained in [24] in our notation for  $\alpha \geq 2/3$ , it is a function of this parameter  $\lambda$  and is thus not plotted here.

## VI. GAUSSIAN TWO-WAY INTERFERENCE CHANNEL

All previous channel models considered were deterministic. We now ask whether we may obtain insight into whether adaptation is useless / useful in certain noisy channels. We do so by considering the Gaussian two-way interference channel, and ask when non-adaptive schemes such as the celebrated Han and Kobayashi [27] perform as well, or nearly as well, as adaptive schemes.

We do not construct any inner bounds which employ adaptation; our focus is on showing when non-adaptive schemes perform “well”. Rather, we derive an outer bounds for the Gaussian two-way IC under

full adaptation (all 4 nodes may adapt) and several under partial adaptation (only 2 of the 4 may adapt) constraints. We then show that non-adaptive schemes sometimes achieve the capacity, or at least to within a constant gap of either the fully or partially adaptive schemes. We note that while the converses and the steps are new and exploit carefully chosen genie, when we evaluate these by further outer-bounding our outer-bounds, interestingly, we sometimes re-obtain some of the outer bounds of the interference channel [7] *or* the interference channel with feedback [21]. This in turn is sufficient to achieve capacity to within a constant gap (which we emphasize, sometimes is limited to *partial* adaptation for some of the weak interference regimes but this will be explicitly mentioned when it is the case).

#### A. Channel model, definitions, and partial adaptation lemma

At each channel use, the Gaussian two-way IC is described by the input/output relationships

$$\begin{aligned} Y_1 &= g_{11}X_1 + g_{21}X_2 + g_{41}X_4 + Z_1 \\ Y_2 &= g_{12}X_1 + g_{22}X_2 + g_{32}X_3 + Z_2 \\ Y_3 &= g_{23}X_2 + g_{33}X_3 + g_{43}X_4 + Z_3 \\ Y_4 &= g_{14}X_1 + g_{34}X_3 + g_{44}X_4 + Z_4, \end{aligned}$$

where  $g_{jk}$ , for  $j, k \in \{1, 2, 3, 4\}$  are the complex channel gains. We assume the power constraints  $E[|X_j|^2] \leq P_j = 1, j \in \{1, 2, 3, 4\}$ , and independent, identically distributed complex Gaussian noise  $Z_j \sim \mathcal{CN}(0, 1)$  at all nodes  $j \in (1, 2, 3, 4)$ , which may be done without loss of generality. Furthermore, we define  $\text{SNR}_{12} = |g_{12}|^2, \text{SNR}_{21} = |g_{21}|^2, \text{SNR}_{34} = |g_{34}|^2, \text{SNR}_{43} = |g_{43}|^2$ , and  $\text{INR}_{14} = |g_{14}|^2, \text{INR}_{41} = |g_{41}|^2, \text{INR}_{23} = |g_{23}|^2, \text{INR}_{32} = |g_{32}|^2$ . Note that we have kept the “self-interference” terms such as  $g_{11}X_1$  in the expression of  $Y_1$  (for example). In this Gaussian model, it is clear that since node 1 knows  $X_1$  we may equivalently remove this self-interference term due to the additive nature of the channel and hence including it is unnecessary. However, we leave it in our expressions to emphasize precisely this fact. In other channels such as the two-way binary multiplier channel, where  $Y = X_1X_2$  one cannot “undo” ones’ own channel, which is one source of difficulty for this elusive two-way channel. In all converses, the fact that we can cancel or subtract out a node’s “self-interference” is shown explicitly. This is one of the reasons two-way channels of this form, as seen for example in the Gaussian two-way channel as well [5], are easier to deal with, which we hope to emphasize.

We say that the Gaussian two-way interference channel operates under “full adaptation” if we allow

$$X_{1,i} = f_1(M_{12}, Y_1^{i-1}), \quad X_{2,i} = f_2(M_{21}, Y_2^{i-1}) \quad (19)$$

$$X_{3,i} = f_3(M_{34}, Y_3^{i-1}), \quad X_{4,i} = f_4(M_{43}, Y_4^{i-1}). \quad (20)$$



Similarly, it operates under “partial adaptation” if we only allow the following:

$$X_{1,i} = f_1(M_{12}), \quad X_{2,i} = f_2(M_{21}, Y_2^{i-1}) \quad (21)$$

$$X_{3,i} = f_3(M_{34}), \quad X_{4,i} = f_4(M_{43}, Y_4^{i-1}), \quad (22)$$

i.e. nodes 1 and 3 are “restricted” [3]. By symmetry, we may alternatively allow nodes 2 and 4 to be restricted and 1, 3 to be fully adaptive; whether allowing 1, 2 or 1, 4 to be restricted and the complement fully adaptive remains an open problem.

We are interested in the symmetric capacity (or sum-rate), when all the SNRs equal a given SNR, and all the INRs equal a given INR. For full adaptation, due to the symmetry, we consider the per-user rates  $R_{sym} = \frac{R_{12}+R_{34}}{2} = \frac{R_{21}+R_{43}}{2}$ . In partial adaptation, there is only partial symmetry (nodes 1 and 3 are fixed, while 2 and 4 are not), and hence we will consider the per user rates  $R_{sym \rightarrow} = \frac{R_{12}+R_{34}}{2}$  and  $R_{sym \leftarrow} = \frac{R_{21}+R_{43}}{2}$  for the forward and reverse directions respectively. We will derive outer bounds for  $R_{sym}$  under full adaptation and  $R_{sym \rightarrow}, R_{sym \leftarrow}$  under partial adaptation, and show these to be achievable to within constant gaps by non-adaptive schemes.

We first prove a modified version of Lemma 8 relevant in partial adaptation for the Gaussian channel.

*Lemma 10:* Under partial adaptation (21) – (22), for some deterministic functions  $f_5$  and  $f_6$ ,

$$X_{2,i} = f_5(M_{12}, M_{21}, M_{34}, Z_2^{i-1}) \perp M_{43}, \quad \forall i \quad (23)$$

$$X_{4,i} = f_6(M_{43}, M_{34}, M_{12}, Z_4^{i-1}) \perp M_{21}, \quad \forall i \quad (24)$$

where  $\perp$  denotes independence.

*Proof:* Note that  $X_{2,i} = f_2(M_{21}, Y_2^{i-1})$  and  $Y_2^{i-1} = g_{12}X_1^{i-1} + g_{22}X_2^{i-1} + g_{32}X_3^{i-1} + Z_2^{i-1}$ . Since  $X_1^{i-1}$  and  $X_3^{i-1}$  are functions only of  $M_{12}$  and  $M_{34}$  respectively, we may conclude that there exists a function  $f^*$  such that  $X_{2,i} = f^*(M_{21}, M_{12}, M_{34}, X_2^{i-1}, Z_2^{i-1})$ . Iterating this argument, and noting that  $X_{2,1}$  is only a function of  $M_{21}$ , we obtain the lemma. The result for  $X_{4,i}$  follows similarly. That  $X_{2,i}$  is independent of  $M_{43}$  follows since  $M_{43}$  is independent of all the arguments inside  $f^*$ . ■

## B. Outer bounds

We now present two outer bounds for the Gaussian two-way IC under full and partial adaptation respectively. These bounds are either within a constant gap, or sufficient to show the capacity depending on different regimes. We will derive general outer bounds, imposing symmetry only in the final step.

*Theorem 11: Outer bound: full adaptation.* For the Gaussian two-way symmetric IC under full adaptation, any achievable symmetric rate  $R_{sym} = \frac{R_{12}+R_{34}}{2} = \frac{R_{21}+R_{43}}{2}$ , achievable by each user, satisfies,

$$R_{sym} \leq \frac{1}{2} \log(1 + \text{SNR} + \text{INR} + 2\sqrt{\text{SNR} \times \text{INR}}) + \frac{1}{2} \log\left(1 + \frac{\text{SNR}}{1 + \text{INR}}\right) \quad (25)$$

*Proof:*

It is sufficient to consider  $R_{12} + R_{34}$  due to symmetry. This bound is inspired by the corresponding sum-rate bound in the linear deterministic model, i.e., we add asymmetric genie  $Y_2^n$  at node 4. Notice the genie  $Z_1^n$  in the conditioning of both terms as well.

$$\begin{aligned}
& n(R_{12} + R_{34} - \epsilon) \\
& \leq I(M_{12}; Y_2^n | M_{21}, M_{43}, Z_1^n) + I(M_{34}; Y_4^n, Y_2^n | M_{12}, M_{21}, M_{43}, Z_1^n) \\
& \leq I(M_{12}; Y_2^n | M_{21}, M_{43}, Z_1^n) + I(M_{34}; Y_2^n | M_{21}, M_{12}, M_{43}, Z_1^n) + H(Y_4^n | M_{21}, M_{12}, M_{43}, Y_2^n, Z_1^n) - H(Z_4^n) \\
& \stackrel{(a)}{=} I(M_{12}; Y_2^n | M_{21}, M_{43}, Z_1^n) + I(M_{34}; Y_2^n | M_{21}, M_{12}, M_{43}, Z_1^n) \\
& + \sum_{i=1}^n [H(g_{34}X_{3,i} + Z_{4,i} | M_{21}, M_{12}, M_{43}, Y_4^{i-1}, X_4^i, Y_2^n, X_2^n, Z_1^n, X_1^i)] - H(Z_4^n) \\
& \stackrel{(b)}{\leq} \sum_{i=1}^n [H(Y_{2,i} | Y_2^{i-1}, M_{21}, X_{2,i}) - H(Y_{2,i} | Y_2^{i-1}, M_{12}, M_{21}, M_{43}, Z_1^n) \\
& + H(Y_{2,i} | Y_2^{i-1}, M_{12}, M_{21}, M_{43}, Z_1^n) - H(Z_{2,i}) + H(g_{34}X_{3,i} + Z_{4,i} | X_{4,i}, g_{32}X_{3,i} + Z_{2,i}, X_1^i, X_2^n) - H(Z_{4,i})] \\
& \stackrel{(c)}{\leq} \sum_{i=1}^n H(g_{12}X_{1,i} + g_{32}X_{3,i} + Z_{2,i} | X_{2,i}) - H(Z_{2,i}) + H(g_{34}X_{3,i} + Z_{4,i} | X_{4,i}, g_{32}X_{3,i} + Z_{2,i}) - H(Z_{4,i})
\end{aligned} \tag{26}$$

In step (a),  $X_1^i$  in the conditioning of the third term is constructed from  $(M_{12}, X_2^n, X_4^i, Z_1^n)$ . In step (b), we used conditioning reduces entropy, the second and the third term cancelled each other and  $g_{32}X_{3,i} + Z_{2,i}$  in the conditioning of the fifth term is decoded from  $Y_2^n$ . In step (c), we only keep the self-interference  $X_{4,i}$  and drop the terms  $X_1^i, X_2^n$  in the conditioning of the third term. We could leave these and express the outer bound in terms of correlation coefficients between the inputs (which in general may be correlated due to full adaptation). However, in subsequent steps we will seek to maximize, or outer bound this outer bound to obtain a simple analytical expression, which amounts to setting certain correlation coefficients to 0, or equivalently, dropping the terms  $X_1^i, X_2^n$  in the conditioning. Further evaluation yields (25), for details please refer to Appendix VIII-E. ■

*Remark 10: Sum-rate bound:* Note that the final, evaluated symmetric, normalized sum-rate bound in (25) has the same form as the IC with perfect output feedback outer bound [21, upper bound on (7)], though they are arrived at using different genies (though similar in the sense that the genies given to the destinations are asymmetric). In both channel models, inputs may be arbitrarily correlated (unless one would make further arguments restricting the input distributions), leading to similar bounding techniques using correlation coefficients.

*Theorem 12: Outer bound: partial adaptation.* For the Gaussian two-way IC under partial adaptation (21) – (22), in addition to the bounds in Theorem 11, we may also conclude that any achievable rates  $(R_{12}, R_{21}, R_{34}, R_{43})$ , and  $R_{sym\rightarrow} = \frac{R_{12}+R_{34}}{2}$  and  $R_{sym\leftarrow} = \frac{R_{21}+R_{43}}{2}$  must satisfy,

$$R_{12} \leq \log(1 + \text{SNR}_{12}) \quad (27)$$

$$R_{21} \leq \log(1 + \text{SNR}_{21}) \quad (28)$$

$$R_{34} \leq \log(1 + \text{SNR}_{34}) \quad (29)$$

$$R_{43} \leq \log(1 + \text{SNR}_{43}) \quad (30)$$

$$R_{sym\rightarrow} \leq \log \left( 1 + \text{INR} + \text{SNR} - \frac{\text{INR} \times \text{SNR}}{1 + \text{INR}} \right) \quad (31)$$

$$R_{sym\leftarrow} \leq \begin{cases} \log \left( 1 + \text{INR} + \frac{\text{SNR}}{\text{INR}} \right), & \text{if } \text{SNR} \leq \text{INR}^3 \\ \log \left( 1 + \frac{(\sqrt{\text{SNR}} + \sqrt{\text{INR}})^2}{1 + \text{INR}} \right), & \text{if } \text{SNR} > \text{INR}^3 \end{cases} \quad (32)$$

*Proof:* For the single-rate bounds, it is sufficient to show the first two due to symmetry (notice that we must treat the  $\rightarrow$  and  $\leftarrow$  directions separately however due to the asymmetry of the partial adaptation problem definition).

$$\begin{aligned} n(R_{12} - \epsilon) &\leq I(M_{12}; Y_2^n | M_{21}, M_{34}) \\ &\leq H(Y_2^n | M_{21}, M_{34}) - H(Y_2^n | M_{21}, M_{34}, M_{12}, X_1^n, X_2^n, X_3^n) \\ &\stackrel{(a)}{\leq} \sum_{i=1}^n [H(Y_{2,i} | Y_2^{i-1}, M_{21}, X_{2,i}, M_{34}, X_{3,i}) - H(Z_{2,i})] \\ &\leq \sum_{i=1}^n [H(g_{12}X_{1,i} + Z_{2,i}) - H(Z_{2,i})] \\ &\leq \sum_{i=1}^n [\log(1 + \text{SNR}_{12})] \\ n(R_{21} - \epsilon) &\leq I(M_{21}; Y_1^n | M_{12}, M_{43}, M_{34}, Z_4^{n-1}) \\ &\leq H(Y_1^n | M_{12}, M_{34}, M_{43}, Z_4^{n-1}) - H(Y_1^n | M_{12}, M_{34}, M_{43}, Z_4^{n-1}, M_{21}, X_1^n, X_2^n, X_4^n) \\ &\stackrel{(b)}{\leq} \sum_{i=1}^n [H(Y_{1,i} | M_{12}, M_{34}, M_{43}, Z_4^{n-1}, Y_1^{i-1}, X_{1,i}, X_{4,i}) - H(Z_{1,i})] \\ &\leq \sum_{i=1}^n [H(g_{21}X_{2,i} + Z_{1,i}) - H(Z_{1,i})] \\ &\leq \sum_{i=1}^n [\log(1 + \text{SNR}_{21})] \end{aligned}$$

where (a) follows from the definition of partial adaptation and (b) follows by the same reason, as well as Lemma 10.

For the  $\rightarrow$  direction of the symmetric rate,

$$\begin{aligned}
n(R_{12} + R_{34} - \epsilon) &\leq I(M_{12}; Y_2^n, g_{14}X_1^n + Z_4^n, M_{21}, M_{43}) + I(M_{34}; Y_4^n, g_{32}X_3^n + Z_2^n, M_{21}, M_{43}) \\
&\stackrel{(a)}{=} H(Y_2^n | g_{14}X_1^n + Z_4^n, M_{43}, M_{21}) + H(g_{14}X_1^n + Z_4^n | M_{43}, M_{21}) - H(Y_2^n, g_{14}X_1^n + Z_4^n | M_{12}, M_{21}, M_{43}) \\
&\quad + H(Y_4^n | g_{32}X_3^n + Z_2^n, M_{43}, M_{21}) + H(g_{32}X_3^n + Z_2^n | M_{43}, M_{21}) - H(Y_4^n, g_{32}X_3^n + Z_2^n | M_{34}, M_{21}, M_{43}) \\
&\stackrel{(b)}{=} H(Y_2^n | g_{14}X_1^n + Z_4^n, M_{43}, M_{21}) + \sum_{i=1}^n [H(g_{14}X_{1,i} + Z_{4,i} | g_{14}X_1^{i-1} + Z_4^{i-1}, M_{43}, M_{21}, M_{34}) \\
&\quad - H(Y_{2,i}, g_{14}X_{1,i} + Z_{4,i} | Y_2^{i-1}, g_{14}X_1^{i-1} + Z_4^{i-1}, M_{12}, M_{21}, M_{43}, X_2^i, X_1^i)] \\
&\quad + H(Y_4^n | g_{32}X_3^n + Z_2^n, M_{43}, M_{21}) + \sum_{i=1}^n [H(g_{32}X_{3,i} + Z_{2,i} | g_{32}X_3^{i-1} + Z_2^{i-1}, M_{43}, M_{21}, M_{12}) \\
&\quad - H(Y_{4,i}, g_{32}X_{3,i} + Z_{2,i} | Y_4^{i-1}, g_{32}X_3^{i-1} + Z_2^{i-1}, M_{34}, M_{21}, M_{43}, X_4^i, X_3^i)] \\
&\stackrel{(c)}{=} H(Y_2^n | g_{14}X_1^n + Z_4^n, M_{43}, M_{21}) + \sum_{i=1}^n [H(g_{14}X_{1,i} + Z_{4,i} | g_{14}X_1^{i-1} + Z_4^{i-1}, M_{43}, M_{21}, M_{34}, X_3^i, Y_4^{i-1}, X_4^i, \\
&\quad g_{32}X_3^{i-1} + Z_2^{i-1}) - H(g_{32}X_{3,i} + Z_{2,i}, Z_{4,i} | Y_2^{i-1}, g_{14}X_1^{i-1} + Z_4^{i-1}, M_{12}, M_{21}, M_{43}, X_2^i, X_1^i, g_{32}X_3^{i-1} + Z_2^{i-1})] \\
&\quad + H(Y_4^n | g_{32}X_3^n + Z_2^n, M_{43}, M_{21}) + \sum_{i=1}^n [H(g_{32}X_{3,i} + Z_{2,i} | g_{32}X_3^{i-1} + Z_2^{i-1}, M_{43}, M_{21}, M_{12}, X_1^i, Y_2^{i-1}, X_2^i, \\
&\quad g_{14}X_1^{i-1} + Z_4^{i-1}) - H(g_{14}X_{1,i} + Z_{4,i}, Z_{2,i} | Y_4^{i-1}, g_{32}X_3^{i-1} + Z_2^{i-1}, M_{34}, M_{21}, M_{43}, X_4^i, X_3^i, g_{14}X_1^{i-1} + Z_4^{i-1})] \\
&\stackrel{(d)}{=} \sum_{i=1}^n [H(Y_{2,i} | Y_2^{i-1}, g_{14}X_1^n + Z_4^n, M_{43}, M_{21}) - H(Z_{2,i}) + H(Y_{4,i} | Y_4^{i-1}, g_{32}X_3^n + Z_2^n, M_{43}, M_{21}) - H(Z_{4,i})] \\
&\leq \sum_{i=1}^n [H(g_{12}X_{1,i} + g_{32}X_{3,i} + Z_{2,i} | g_{14}X_{1,i} + Z_{4,i}, X_{2,i}) - H(Z_{2,i}) \\
&\quad + H(g_{34}X_{3,i} + g_{14}X_{1,i} + Z_{4,i} | g_{32}X_{3,i} + Z_{2,i}, X_{4,i}) - H(Z_{4,i})] \tag{33}
\end{aligned}$$

In the first step, we have given  $(g_{14}X_1^n + Z_4^n)$  and  $(g_{32}X_3^n + Z_2^n)$  as side information. Step (a) follows from the independence of the messages. In step (b), the 2nd and 5th terms follow since  $g_{14}X_{1,i}$  and  $g_{32}X_{3,i}$  are functions only of  $M_{12}$  and  $M_{34}$ , and the 3rd and 6th terms follow from the definition of partial adaptation. For (c), in the conditioning of the 2nd term, we are able to add  $(X_3^i, g_{32}X_3^{i-1} + Z_2^{i-1})$  due to partial adaptation constraints, and  $(Y_4^{i-1}, X_4^i)$  are constructed from  $(g_{14}X_1^{i-1} + Z_4^{i-1}, M_{43}, X_3^i)$ . The 5th term follows similarly. In step (d),  $-H(Z_{2,i})$  and  $-H(Z_{4,i})$  are obtained from a portion of the 6th and 3rd terms in (c) respectively using the chain rule (noises are independent from other terms), and the remainder (chain rule) of the 6th and 3rd terms are cancelled by the 2nd and 5th terms respectively.

To obtain (31) we continue to outer bound (33) in terms of SNR and INR, using the fact that Gaussians maximize entropy subject to variance constraints. Specifically, one may intuitively see that, if one defines  $\lambda_{jk} = E[X_j X_k^*]$ , that one may express (33) in terms of  $\lambda_{12}, \lambda_{13}, \lambda_{14}, \lambda_{34}, \lambda_{23}$ . One also notices from the conditional entropy expression in (33) that taking  $\lambda_{14} = \lambda_{23} = \lambda_{12} = \lambda_{34} = 0$ , and since  $\lambda_{13} = 0$

(naturally, by partial adaptation) will maximize the outer bound. This may alternatively be worked out by calculating the conditional covariance matrices directly (as we will show for the next bound on  $R_{\leftarrow}$ ). In this case then, for each  $i$ , we may bound

$$\begin{aligned} H(g_{12}X_1 + g_{32}X_3 + Z_2|g_{14}X_1 + Z_4, X_2) - H(Z_2) &\leq H(g_{12}X_1 + g_{32}X_3 + Z_2|g_{14}X_1 + Z_4) - H(Z_2) \\ &\leq \log 2\pi e(\text{Var}(g_{12}X_1 + g_{32}X_3 + Z_2|g_{14}X_1 + Z_4)) - \log 2\pi e(\text{Var}(Z_2)) \\ &\leq \log \left( 1 + \text{SNR} + \text{INR} - \frac{\text{SNR} \times \text{INR}}{1 + \text{INR}} \right), \end{aligned}$$

which together with the symmetric expressions for the third and fourth terms in (33) yield (31).

For the  $\leftarrow$  direction, we are similarly able to obtain:

$$\begin{aligned} n(R_{21} + R_{43} - \epsilon) &\leq I(M_{21}; Y_1^n, g_{23}X_2^n + Z_3^n, M_{12}, M_{34}) + I(M_{43}; Y_3^n, g_{41}X_4^n + Z_1^n, M_{12}, M_{34}) \\ &= H(Y_1^n|g_{23}X_2^n + Z_3^n, M_{34}, M_{12}) + H(g_{23}X_2^n + Z_3^n|M_{34}, M_{12}) - H(Y_1^n, g_{23}X_2^n + Z_3^n|M_{21}, M_{12}, M_{34}) \\ &\quad + H(Y_3^n|g_{41}X_4^n + Z_1^n, M_{34}, M_{12}) + H(g_{41}X_4^n + Z_1^n|M_{34}, M_{12}) - H(Y_3^n, g_{41}X_4^n + Z_1^n|M_{43}, M_{12}, M_{34}) \\ &\stackrel{(a)}{=} H(Y_1^n|g_{23}X_2^n + Z_3^n, M_{34}, M_{12}) + \sum_{i=1}^n [H(g_{23}X_{2,i} + Z_{3,i}|g_{23}X_2^{i-1} + Z_3^{i-1}, M_{34}, M_{12}, M_{43}) \\ &\quad - H(Y_{1,i}, g_{23}X_{2,i} + Z_{3,i}|Y_1^{i-1}, g_{23}X_2^{i-1} + Z_3^{i-1}, M_{21}, M_{12}, M_{34}, X_1^i, Z_2^{i-1}, X_2^i)] \\ &\quad + H(Y_3^n|g_{41}X_4^n + Z_1^n, M_{34}, M_{12}) + \sum_{i=1}^n [H(g_{41}X_{4,i} + Z_{1,i}|g_{41}X_4^{i-1} + Z_1^{i-1}, M_{34}, M_{12}, M_{21}) \\ &\quad - H(Y_{3,i}, g_{41}X_{4,i} + Z_{1,i}|Y_3^{i-1}, g_{41}X_4^{i-1} + Z_1^{i-1}, M_{43}, M_{12}, M_{34}, X_3^i, Z_4^{i-1}, X_4^i)] \\ &= H(Y_1^n|g_{23}X_2^n + Z_3^n, M_{34}, M_{12}) + \sum_{i=1}^n [H(g_{23}X_{2,i} + Z_{3,i}|g_{23}X_2^{i-1} + Z_3^{i-1}, M_{34}, M_{12}, M_{43}, Z_4^{i-1}, g_{41}X_4^{i-1} + Z_1^{i-1}, \\ &\quad X_3^i, X_4^i, Y_3^{i-1}) - H(g_{41}X_{4,i} + Z_{1,i}, Z_{3,i}|Y_1^{i-1}, g_{23}X_2^{i-1} + Z_3^{i-1}, M_{21}, M_{12}, M_{34}, Z_2^{i-1}, X_1^i, X_2^i, g_{41}X_4^{i-1} + Z_1^{i-1})] \\ &\quad + H(Y_3^n|g_{41}X_4^n + Z_1^n, M_{34}, M_{12}) + \sum_{i=1}^n [H(g_{41}X_{4,i} + Z_{1,i}|g_{41}X_4^{i-1} + Z_1^{i-1}, M_{34}, M_{12}, M_{21}, Z_2^{i-1}, g_{23}X_2^{i-1} + Z_3^{i-1}, \\ &\quad X_1^i, X_2^i, Y_1^{i-1}) - H(g_{23}X_{2,i} + Z_{3,i}, Z_{1,i}|Y_3^{i-1}, g_{41}X_4^{i-1} + Z_1^{i-1}, M_{43}, M_{12}, M_{34}, Z_4^{i-1}, X_3^i, X_4^i, g_{23}X_2^{i-1} + Z_3^{i-1})] \\ &= \sum_{i=1}^n [H(Y_{1,i}|Y_1^{i-1}, g_{23}X_2^n + Z_3^n, M_{34}, M_{12}) - H(Z_{1,i}) + H(Y_{3,i}|Y_3^{i-1}, g_{41}X_4^n + Z_1^n, M_{34}, M_{12}) - H(Z_{3,i})] \\ &\leq \sum_{i=1}^n [H(g_{21}X_{2,i} + g_{41}X_{4,i} + Z_{1,i}|g_{23}X_{2,i} + Z_{3,i}, X_{1,i}) - H(Z_{1,i}) \\ &\quad + H(g_{43}X_{4,i} + g_{23}X_{2,i} + Z_{3,i}|g_{41}X_{4,i} + Z_{1,i}, X_{3,i}) - H(Z_{3,i})] \end{aligned} \tag{34}$$

The slight differences compared to the previous outer bound are: in step (a), we add  $M_{43}$  in the conditioning part of the second term since it is independent of  $X_{2,i}$  according to Lemma 10; in the conditioning part

of the third term we add  $Z_2^{i-1}$  (independent), which, together with  $(M_{12}, M_{21}, M_{34})$  allow us to construct  $X_2^i$ . Similar arguments can be made for the fifth and sixth term.

We again proceed to outer bound (34) to obtain (32). Again, it is sufficient to evaluate the first two terms in (34) due to symmetry. Once again, we could outer bound (34) in terms of the conditional covariance matrices and then proceed to select values of the correlation coefficients (complex)  $\lambda_{ij} := E[X_i X_j^*]$  which maximize this outer bound. A more intuitive method is to note that again, the conditional entropies in (34) will be maximized if  $\lambda_{14} = \lambda_{32} = 0$ , and  $\lambda_{12} = \lambda_{34} = 0$  (similar to (40)), which may also be obtained by dropping  $X_{1,i}, X_{3,i}$  in the conditioning terms. At that point, we are only left with the coefficient  $\lambda_{24} = E[X_2 X_4^*]$ , (which in contrast to the  $\rightarrow$  bound is not automatically 0 due to the possible adaptation in the  $\leftarrow$  direction. Furthermore, setting it to zero cannot be argued intuitively as we see a tradeoff.) yielding the following bound for  $R_{sym\leftarrow} = \frac{R_{21}+R_{43}}{2}$  by symmetry:

$$\begin{aligned}
R_{sym\leftarrow} &\leq H(g_{21}X_2 + g_{41}X_4 + Z_1|g_{23}X_2 + Z_3, X_1) - H(Z_1) \\
&\leq H(g_{21}X_2 + g_{41}X_4 + Z_1|g_{23}X_2 + Z_3) - H(Z_1) \\
&\leq \log 2\pi e (\text{Var}(g_{21}X_2 + g_{41}X_4 + Z_1|g_{23}X_2 + Z_3)) - \log 2\pi e (\text{Var}(Z_1)) \\
&\leq \log \left( 1 + \text{INR} + \text{SNR} + 2|\lambda_{24}| \cos \theta \sqrt{\text{SNR} \times \text{INR}} - \frac{\text{SNR} \times \text{INR} + \text{INR}^2 |\lambda_{24}|^2 + 2\sqrt{\text{SNR} \text{INR}^3} |\lambda_{24}| \cos \theta}{1 + \text{INR}} \right).
\end{aligned} \tag{35}$$

where  $\theta$  is the angle of  $g_{21}g_{41}^* \lambda_{24}$ . To maximize (35), we take the partials of the expression with respect to  $|\lambda_{24}|$  and  $\theta$  and set these to 0. For these to equal 0 for all SNR and INR we must have  $\theta = 0$  and  $|\lambda_{24}| = \frac{\sqrt{\text{SNR} \times \text{INR}}}{\text{INR}^2}$  (discussed next). Note that we must constrain  $|\lambda_{24}| \in [0, 1]$ . In the interval  $|\lambda_{24}| \in \left[0, \frac{\sqrt{\text{SNR} \times \text{INR}}}{\text{INR}^2}\right]$  one may verify that the function is increasing in  $|\lambda_{24}|$ . Thus, if  $\frac{\sqrt{\text{SNR} \times \text{INR}}}{\text{INR}^2} \leq 1$ , ( $|\lambda_{24}| = \frac{\sqrt{\text{SNR} \times \text{INR}}}{\text{INR}^2}, \theta = 0$ ) maximizes (35); this happens if  $\text{SNR} \leq \text{INR}^3$ , and yields the first bound in (32). Otherwise, for  $\text{SNR} > \text{INR}^3$ , ( $\lambda_{24} = 1, \theta = 0$ ) maximizes (35), yielding the second equation in (32). ■

*Remark 11:* The sum-rate bound for  $R_{sym\rightarrow}$  of (31) has the same form as Etkin, Tse and Wang's outer bound for one-way Gaussian interference channel [7, (12)] which is useful in weak interference. The sum-rate bound for  $R_{sym\leftarrow}$  is quite different, and we note that it may be verified that (32) is always at least as large as (31), as one might expect given the partial adaptation constraints on nodes in the  $\rightarrow$  direction, but none on the nodes in the  $\leftarrow$  direction.

We next show that these outer bounds, derived for the fully adaptive or partially adaptive models, may be achieved to within a constant gap or capacity by *non-adaptive* schemes – i.e. the simultaneous decoding or the Han and Kobayashi scheme operating in the two directions independently. We break our analysis

into three sub-sections: 1) very strong interference, 2) strong interference, and 3) weak interference. The overall finite gap results are summarized in Table I.

*C. Very Strong Interference:*  $\text{INR} \geq \text{SNR}(1 + \text{SNR})$

We first show that a non-adaptive scheme may achieve the capacity for the two-way Gaussian IC under a partially adaptive model in very strong interference. For the symmetric two-way Gaussian IC, define “very strong interference” as the class of channels for which  $\text{INR} \geq \text{SNR}(1 + \text{SNR})$ , as in [7, below equation (21)]. It is well known that the capacity region of the one-way Gaussian IC in very strong interference is that of two parallel Gaussian point-to-point channels [28], which may be achieved by having each receiver first decode the interfering signal, treating its own as noise, subtracting off the decoded interference, and decoding its own message. Given that the interference is so strong, this may be done without a rate penalty. We ask whether the same is true for the two-way Gaussian IC with partial adaptation. The answer is affirmative and the capacity region is given by the following theorem:

*Theorem 13:* The capacity region for the two-way Gaussian interference channel with partial adaptation in very strong interference is the set of rate pairs  $(R_{12}, R_{21}, R_{34}, R_{43})$ , such that (27)–(30) are satisfied.

*Proof:* Each node may ignore its ability to adapt, and rather transmit using a  $\mathcal{CN}(0, 1)$  Gaussian random code. Each receiver may cancel its own self-interference, and then proceed to decode first the single interfering term before decoding its own message. This standard non-adaptive scheme may achieve the outer bound in (27)–(30) in Theorem 12. ■

Interestingly, the capacity region of the two-way Gaussian interference channel with partial adaptation in very strong interference, is equivalent to the capacity regions of two one-way Gaussian interference channels with very strong interference in parallel and is achieved using a non-adaptive scheme. This allows us to conclude that *partial* adaptation is useless in this symmetric and very strong interference regime.

*D. Strong Interference:*  $\text{SNR} \leq \text{INR} \leq \text{SNR}(1 + \text{SNR})$

In this regime, we are able to show that a non-adaptive scheme may achieve capacity to within a constant gap of any *fully* adaptive scheme (in contrast to any *partially* adaptive scheme in the last subsection). A symmetric two-way Gaussian IC, as in [7], is said to be in “strong interference” when  $\text{INR} \geq \text{SNR}$ .

The capacity region of one-way Gaussian interference channel in strong interference is given by [29], and for symmetric channels, the capacity region when the interference is strong but not very strong, i.e.  $\text{SNR} \leq \text{INR} \leq \text{SNR}(1 + \text{SNR})$ , may be written as

$$R_{\text{sym}} = \frac{R_{12} + R_{34}}{2} \leq \frac{1}{2} \log(1 + \text{SNR} + \text{INR}). \quad (36)$$

We note that this rate is achievable for the two-way Gaussian IC by using the simultaneous non-unique decoding scheme for the interference channel in strong interference [16], [29], [30]) in the  $\rightarrow$  and  $\leftarrow$  directions, and noting that any self-interference may be canceled. This is a non-adaptive scheme.

We will show that this non-adaptive scheme which achieves (36) in each direction (i.e.  $R_{sym} = (36)$ ) also achieves to within 1 bit (per user, per direction) of our fully adaptive outer bound (25) in strong but not very strong interference.

*Theorem 14:* The capacity region for two-way symmetric Gaussian interference channel with full adaptation in strong (but not very strong) interference is within 1 bit to (36) (per user, per direction)

*Proof:*

$$\begin{aligned}
 (25) - (36) &= \frac{1}{2} \log(1 + \text{SNR} + \text{INR} + 2\sqrt{\text{SNR} \times \text{INR}}) + \frac{1}{2} \log \left( 1 + \frac{\text{SNR}}{1 + \text{INR}} \right) - \frac{1}{2} \log(1 + \text{SNR} + \text{INR}) \\
 &\stackrel{(a)}{\leq} \frac{1}{2} \log 2(1 + \text{SNR} + \text{INR}) + \frac{1}{2} \log \left( 1 + \frac{\text{SNR}}{1 + \text{INR}} \right) - \frac{1}{2} \log(1 + \text{SNR} + \text{INR}) \\
 &\stackrel{(b)}{\leq} \frac{1}{2} + \frac{1}{2} \log \left( 1 + \frac{\text{INR}}{\text{INR}} \right) \\
 &= 1
 \end{aligned}$$

In step (a), we use the fact that  $1 + \text{SNR} + \text{INR} + 2\sqrt{\text{SNR} \times \text{INR}} \leq 2(1 + \text{SNR} + \text{INR})$ . Step (b) follows from the condition of strong interference  $\text{INR} \geq \text{SNR}$ . Since our bound (25) is valid for the symmetric assumptions of full adaptation, we conclude that the non-adaptive schemes' gap to the fully adaptive outer bound for each user, for each direction is at most 1 bit. ■

*Remark 12:* Note that if we were to evaluate the fully adaptive outer bound of (41) under partial adaptation constraints instead, i.e.,  $X_1$  and  $X_3$  are only functions of  $M_{12}$  and  $M_{34}$  respectively, then we would be able to set  $\lambda_{13}$  in (41) equal to 0, yielding a new outer bound  $R_{sym \rightarrow} \leq \frac{1}{2} \log(1 + \text{SNR} + \text{INR}) + \frac{1}{2} \log \left( 1 + \frac{\text{SNR}}{1 + \text{INR}} \right)$ . In this case a gap of  $\frac{1}{2}$  bit instead of 1 bit maybe shown for  $R_{sym \rightarrow}$ . However, due to the asymmetry of partial adaptation ( $\lambda_{24}$  in general not equal to 0), in the opposite direction, we would still have a 1 bit gap for  $R_{sym \leftarrow}$ .

#### E. Weak Interference: $\text{INR} \leq \text{SNR}$

We now show that the well known Han and Kobayashi scheme employed in parallel in the  $\rightarrow$  and  $\leftarrow$  directions may achieve to within a constant number of bits of the fully or partially adaptive (depends on the channel regimes, or relative SNR and INR values) capacity region for the two-way Gaussian IC.

*Theorem 15:* A non-adaptive scheme may achieve to within a 2 bit per user per direction of partially adaptive capacity region for the two-way Gaussian IC in weak interference. In some channel regimes, this non-adaptive scheme also achieves to within a constant gap of any *fully* adaptive scheme.

*Proof:* As for the one-way IC [7], we break our proof into two regimes:  $\text{INR} \geq 1$  or  $\text{INR} < 1$ .



1)  $\text{INR} \geq 1$ : Outer bounds have already been derived. Consider now using the specific choice of the Han and Kobayashi (HK) strategy utilized for the symmetric one-way IC as in [7, (4)] in each direction. That is, view nodes 1,2 as transmitters and 3,4 as receivers in the  $\rightarrow$  direction and employ the particular choice of the HK scheme where private messages are encoded at the level of the noise, and similarly for the  $\leftarrow$  direction consider nodes 3,4 as transmitters and 1,2 as receivers. Due to the additive nature of the channel and each node's ability to first cancel out their self-interference, one may achieve the following rates per user, per node for each direction when  $\text{INR} \geq 1$  for the symmetric two-way Gaussian IC:

$$R_{HK} = \min \left\{ \frac{1}{2} \log(1 + \text{INR} + \text{SNR}) + \frac{1}{2} \log \left( 2 + \frac{\text{SNR}}{\text{INR}} \right) - 1, \log \left( 1 + \text{INR} + \frac{\text{SNR}}{\text{INR}} \right) - 1 \right\} \quad (37)$$

$$=: \min \{ R_{HK1}, R_{HK2} \}. \quad (38)$$

**If the first term in (37) is active** we show a constant gap to the outer bound (25),

$$\begin{aligned} (25) - R_{HK1} &= \frac{1}{2} \log(1 + \text{SNR} + \text{INR} + 2\sqrt{\text{SNR} \times \text{INR}}) + \frac{1}{2} \log \left( 1 + \frac{\text{SNR}}{1 + \text{INR}} \right) \\ &\quad - \frac{1}{2} \log(1 + \text{INR} + \text{SNR}) - \frac{1}{2} \log \left( 2 + \frac{\text{SNR}}{\text{INR}} \right) + 1 \\ &\leq \frac{1}{2} \log 2(1 + \text{SNR} + \text{INR}) - \frac{1}{2} \log(1 + \text{INR} + \text{SNR}) + \frac{1}{2} \log \left( 1 + \frac{\text{SNR}}{\text{INR}} \right) - \frac{1}{2} \log \left( 2 + \frac{\text{SNR}}{\text{INR}} \right) + 1 \\ &\leq \frac{1}{2} \log(2) + \frac{1}{2} \log(1) + 1 \\ &= 1.5 \end{aligned}$$

*Remark 13:* Since our bound (25) is derived assuming full adaptation, we may conclude that this gap holds for both  $R_{\text{sym} \rightarrow}$  and  $R_{\text{sym} \leftarrow}$  (i.e. holds for  $R_{\text{sym}}$ ). If we were to consider partial adaptation ( $\lambda_{13} = 0$ ), this gap could be reduced to 1 bit instead of 1.5 bits for  $R_{\text{sym} \rightarrow}$ , would remain 1.5 bits for  $R_{\text{sym} \leftarrow}$  as  $\lambda_{24} \neq 0$  in general for partial adaptation.

**If the second term in (37) is active**, we use outer bound (31) for the forward direction, to bound the gap for  $R_{\text{sym} \rightarrow}$  as

$$\begin{aligned} (31) - R_{HK2} &= \log \left( 1 + \text{INR} + \text{SNR} - \frac{\text{INR} \times \text{SNR}}{1 + \text{INR}} \right) - \log \left( 1 + \text{INR} + \frac{\text{SNR}}{\text{INR}} \right) + 1 \\ &= \log \left( \frac{(1 + \text{INR})^2 + \text{SNR}}{1 + \text{INR}} \frac{\text{INR}}{\text{INR}(1 + \text{INR}) + \text{SNR}} \right) + 1 \\ &= \log \left( \frac{\text{INR}(1 + \text{INR})^2 + \text{SNR} \times \text{INR}}{\text{INR}(1 + \text{INR})^2 + \text{SNR}(1 + \text{INR})} \right) + 1 \\ &\leq \log(1) + 1 \\ &= 1 \end{aligned}$$

Since our bound (31) has the same form as the ETW bound [7], the capacity of the Gaussian two-way interference channel with partial adaptation in the forward direction is also to within 1 bit of the specific HK rate (37), (38) when  $\text{INR} \geq 1$ .

We use outer bound (32) for the backward direction, to bound the gap for  $R_{\text{sym} \leftarrow}$ , noting that we need to consider both cases separately. If the first term in (32) is relevant ( $\text{SNR} \leq \text{INR}^3$ ):

$$(32) - R_{HK2} = \log \left( 1 + \text{INR} + \frac{\text{SNR}}{\text{INR}} \right) - \log \left( 1 + \text{INR} + \frac{\text{SNR}}{\text{INR}} \right) + 1 = 1$$

If the second term in (32) is relevant ( $\text{SNR} \geq \text{INR}^3$ ):

$$\begin{aligned} & (32) - R_{HK2} \\ &= \log \left( 1 + \frac{(\sqrt{\text{SNR}} + \sqrt{\text{INR}})^2}{1 + \text{INR}} \right) - \log \left( 1 + \text{INR} + \frac{\text{SNR}}{\text{INR}} \right) + 1 \\ &= \log \left( \frac{(1 + 2\text{INR} + \text{SNR} + 2\sqrt{\text{SNR} \times \text{INR}})\text{INR}}{(1 + \text{INR})(\text{INR} + \text{SNR})} \right) + 1 \\ &\stackrel{(a)}{\leq} \log \left( \frac{(2(1 + \text{INR} + \text{SNR}) + \text{INR})\text{INR}}{(1 + \text{INR})(\text{INR} + \text{SNR})} \right) + 1 \\ &= \log \left( \frac{2\text{INR} + 2\text{SNR} \times \text{INR} + 3\text{INR}^2}{\text{INR} + 2\text{INR}^2 + \text{SNR} + \text{INR}^3 + \text{SNR} \times \text{INR}} \right) + 1 \\ &\leq \log \left( \frac{2(\text{INR} + \text{SNR} \times \text{INR} + 2\text{INR}^2 + \text{SNR} + \text{INR}^3)}{\text{INR} + \text{SNR} \times \text{INR} + 2\text{INR}^2 + \text{SNR} + \text{INR}^3} \right) + 1 \\ &= \log(2) + 1 \\ &= 2 \end{aligned}$$

where (a) follows the fact that  $1 + \text{SNR} + \text{INR} + 2\sqrt{\text{SNR} \times \text{INR}} \leq 2(1 + \text{SNR} + \text{INR})$ .

*Remark 14:* We have shown that the capacity region of Gaussian two-way interference channel with partial adaptation (fix  $X_1$  and  $X_3$ ) is within at most 2 bits per user per direction to the region achieved by two simultaneous HK schemes in opposite directions when  $\text{INR} \geq 1$  for both directions. Again, we may conclude that partial adaptation cannot significantly increase the capacity for Gaussian two-way IC with weak interference.

2)  $\text{INR} < 1$ : In this case, a symmetric version of the HK scheme may be obtained from [7, (69)], for which each of the four users may achieve the following rate:

$$R_{\text{INR} < 1} \leq \log \left( 1 + \frac{\text{SNR}}{1 + \text{INR}} \right) \quad (39)$$

Two-way Interference	Constant Gaps per user per direction, in bits (to outer bound)				
Very Strong	0 (partial)				
Strong	1 (full)				
Weak	INR < 1	1 (full)			
	INR ≥ 1	HK1 is active	1.5 (full)		
		HK2 is active	→ direction	1 (partial)	
			← direction	SNR ≤ INR <sup>3</sup>	1 (partial)
				SNR > INR <sup>3</sup>	2 (partial)

TABLE I

CONSTANT GAPS BETWEEN NON-ADAPTIVE SYMMETRIC HAN AND KOBAYASHI SCHEMES IN EACH DIRECTION AND PARTIALLY OR FULLY ADAPTIVE OUTER BOUNDS FOR THE TWO-WAY GAUSSIAN IC.

**We show that this achieves to within 1 bit of the outer bound (25)**

$$\begin{aligned}
(25) - R_{\text{INR} < 1} &\leq \frac{1}{2} \log(1 + \text{SNR} + \text{INR} + 2\sqrt{\text{SNR} \times \text{INR}}) + \frac{1}{2} \log\left(1 + \frac{\text{SNR}}{1 + \text{INR}}\right) - \log\left(1 + \frac{\text{SNR}}{1 + \text{INR}}\right) \\
&= \frac{1}{2} \log(1 + \text{SNR} + \text{INR} + 2\sqrt{\text{SNR} \times \text{INR}}) - \frac{1}{2} \log\left(1 + \frac{\text{SNR}}{1 + \text{INR}}\right) \\
&\leq \frac{1}{2} \log\left(\frac{2(1 + \text{SNR} + \text{INR})(1 + \text{INR})}{1 + \text{SNR} + \text{INR}}\right) \\
&\stackrel{(a)}{\leq} \frac{1}{2} \log(4) \\
&= 1
\end{aligned}$$

where (a) we use the condition of  $\text{INR} < 1$ . Since (25) was obtained for full adaptation, we can conclude that the capacity of the Gaussian two-way interference channel is to within 1 bit to the HK region when  $\text{INR} < 1$  for both directions. ■

We summarize our results for constant gaps in Table I.

## VII. CONCLUSION

In this work, we have demonstrated a few examples of two-way multi-user channels for which adaptation, or the ability of nodes to adapt their current channel inputs based on previously received channel outputs, is useless from a capacity region perspective, i.e. non-adaptive schemes achieve outer bounds derived for the fully adaptive models. Specifically, we obtained the capacity regions of the two-way MAC/BC channel, the two-way Z channel, and the two-way IC of binary modulo-2 addition model, the “deterministic, invertible and cardinality constrained” model, and the linear deterministic model. Interestingly, adaptation (full or partial) is not needed to attain the capacity regions even though it is

permitted. Turning to an example of a noisy channel, we considered the Gaussian two-way IC with 4 terminals and 4 messages. There, it was shown that partial adaptation is useless in very strong interference, and for all other regimes non-adaptive schemes achieved to within constant gaps of fully, or partially, adaptive schemes. We have demonstrated several examples where adaptation is useless – the question of when adaptation is useless in general networks remains a challenging open question. However, based on some of the examples seen here, we believe that the following properties may be needed to make the claim that “adaptation is useless” for a particular network: 1) the self-interference can be cancelled (excludes the binary multiplier channel), 2) no loop in the networks (excludes the relaying of data along stronger paths), and 3) no “coherent” gains (excluding possible gains by having users use adaptation to create joint input distributions in for example Gaussian networks).

## VIII. APPENDIX

### A. Proof of Theorem 1

*Proof:* We first prove the converse; achievability follows via (non-adaptive) time-sharing between nodes in the same direction.

*Proof of bound (2):* The first bound follows from the sum-rate outer bound of the multiple-access channel.

$$\begin{aligned}
 & n(R_{12} + R_{32} - \epsilon) \\
 & \leq I(M_{12}, M_{32}; Y_2^n) = H(Y_2^n) - H(Y_2^n | M_{12}, M_{32}) \\
 & \leq \sum_{i=1}^n [H(Y_{2,i} | Y_2^{i-1})] \leq \sum_{i=1}^n [H(Y_{2,i})] \leq n
 \end{aligned}$$

where the last inequality follows as  $Y_{2,i}$  is a binary random variable whose entropy is thus bounded by 1.

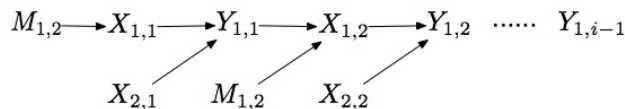


Fig. 4. The Markov chain used in the outer bound proof of Theorem 1.

*Proof of bound (3):*

$$\begin{aligned}
& n(R_{21} + R_{23} - \epsilon) \\
& \stackrel{(a)}{\leq} I(M_{21}; Y_1^n | M_{12}, M_{32}) + I(M_{23}; Y_3^n | M_{12}, M_{32}, M_{21}) \\
& \stackrel{(b)}{\leq} H(Y_1^n | M_{12}, M_{32}) - H(Y_1^n | M_{21}, M_{12}, M_{32}) + H(Y_3^n | M_{12}, M_{32}, M_{21}) \\
& \stackrel{(c)}{\leq} \sum_{i=1}^n [H(Y_{1,i}) - H(Y_{1,i} | M_{21}, M_{12}, M_{32}, Y_1^{i-1}) + H(Y_{3,i} | M_{12}, M_{32}, M_{21}, Y_3^{i-1})] \\
& \stackrel{(d)}{=} \sum_{i=1}^n [H(Y_{1,i}) - H(X_{1,i} \oplus X_{2,i} | M_{21}, M_{12}, M_{32}, Y_1^{i-1}, X_1^i) + H(X_{2,i} \oplus X_{3,i} | M_{12}, M_{32}, M_{21}, Y_3^{i-1}, X_3^i)] \\
& \stackrel{(e)}{\leq} \sum_{i=1}^n [H(Y_{1,i}) - H(X_{2,i} | M_{21}, M_{12}, M_{32}, Y_1^{i-1}, X_1^i, X_2^{i-1}, X_3^i, Y_3^{i-1}) \\
& \quad + H(X_{2,i} | M_{21}, M_{12}, M_{32}, Y_1^{i-1}, X_1^i, X_2^{i-1}, X_3^i, Y_3^{i-1})] \\
& = \sum_{i=1}^n [H(Y_{1,i})] \leq n
\end{aligned}$$

where (a) follows from Fano's inequality, (b) drops a negative entropy term, (c) uses the chain rule and conditioning reduces entropy. In (d),  $X_1^i = f_1(M_{12}, Y_1^{i-1})$  and  $X_3^i = f_3(M_{32}, Y_3^{i-1})$ . We cancel  $X_{1,i}$  and  $X_{3,i}$  in the entropy term in (e) since we know  $X_1^i$  and  $X_3^i$  respectively. In addition, we introduce genes  $X_2^{i-1}, Y_3^{i-1}, X_3^i$  in the negative entropy term. For the third entropy term,  $X_2^{i-1}$  may be obtained from  $Y_3^{i-1} = X_2^{i-1} \oplus X_3^{i-1}$  (bit-wise modulo 2) since we know  $X_3^{i-1}$ . We may obtain  $Y_1^{i-1}$  using the Markov chain illustrated in Fig. 4. Finally,  $X_1^i$  is given by the encoding function  $X_1^i = f_1(M_{12}, Y_1^{i-1})$ . We again bound  $H(Y_{1,i}) \leq 1$ .

We are able to achieve this outer bound using two time-sharing random variables *without adaptation*:  $\alpha$  time-shares between channel inputs  $X_1$  and  $X_3$  for the MAC channel in the  $\rightarrow$  direction, while  $\beta$  time-shares between the messages  $M_{21}$  and  $M_{23}$  (encoded as  $X_2(M_{21})$  and  $X_2(M_{23})$  respectively) in the BC in the  $\leftarrow$  direction. Both directions ignore the received signals and use i.i.d. Bernoulli(1/2) codebooks. ■

### B. Proof of Theorem 2

*Proof:* Outer bound (4) follows by the cut-set bound, exactly as shown for the binary modulo 2 adder MAC/BC. Outer bound (5) may be shown using the cut-set bound, but may alternatively be shown as:

$$\begin{aligned}
& n(R_{21} + R_{23} - \epsilon) \\
& \leq I(M_{21}; Y_1^n | M_{12}, M_{32}) + I(M_{23}; Y_3^n | M_{12}, M_{32}, M_{21}) \\
& \leq H(Y_1^n | M_{12}, M_{32}) - H(Y_1^n | M_{21}, M_{12}, M_{32}) + H(Y_3^n | M_{12}, M_{32}, M_{21}) \\
& \leq \sum_{i=1}^n [H(Y_{1,i}) - H(Y_{1,i} | M_{21}, M_{12}, M_{32}, Y_1^{i-1}) + H(Y_{3,i} | M_{12}, M_{32}, M_{21}, Y_3^{i-1})] \\
& \stackrel{(a)}{=} \sum_{i=1}^n [H(Y_{1,i}) - H(X_{2,i} | M_{21}, M_{12}, M_{32}, Y_1^{i-1}, X_1^i) + H(X_{2,i} | M_{12}, M_{32}, M_{21}, Y_3^{i-1}, X_3^i)] \\
& \stackrel{(b)}{\leq} \sum_{i=1}^n [H(Y_{1,i}) - H(X_{2,i} | M_{21}, M_{12}, M_{32}, Y_1^{i-1}, X_1^i, X_2^{i-1}, X_3^i, Y_3^{i-1}) \\
& \quad + H(X_{2,i} | M_{21}, M_{12}, M_{32}, Y_1^{i-1}, X_1^i, X_2^{i-1}, X_3^i, Y_3^{i-1})] \\
& = \sum_{i=1}^n [H(Y_{1,i})] \leq n \log \kappa
\end{aligned}$$

where (a) follows from the invertibility condition **P2**. In (b),  $X_2^{i-1}$  in the conditioning of the third term is decoded from  $Y_3^{i-1}$  using the invertibility condition **P2**.

Our achievability scheme consists of time-sharing between user 1 and user 3 in the  $\rightarrow$  (MAC) direction, while simultaneously time-sharing between sending data to user 1 and 3 in the  $\leftarrow$  (BC) direction, as in Fig. 5. There, we see two time-sharing coefficients  $0 \leq \alpha, \beta \leq 1$ , where  $\alpha$  time-shares in the  $\rightarrow$  direction and  $\beta$  in the  $\leftarrow$  direction. Let us consider the rates achieved in time slot (1), of duration  $\alpha$  (WLOG we have taken  $\alpha < \beta$ ). Node 1 encodes  $M_{12}$  into  $X_1$  uniformly distributed over the  $\log(\kappa)$  input symbols; node 2 encodes  $M_{21}$  into  $X_2$  uniformly distributed over  $\log(\kappa)$  input symbols and node 3 fixes  $X_3 = x_3^*$  (rate 0). We claim this scheme achieves the rates  $R_{12} = R_{21} = \alpha \log(\kappa)$ ,  $R_{23} = R_{32} = 0$ . Consider  $R_{12}$ : node 2 receives  $Y_2 = F_2(X_1, X_2, x_3^*)$ . Since node 2 knows  $X_2$  and knows that  $X_3 = x_3^*$ , by **P2**, it may construct  $X_1 = G_{21}(X_2, x_3^*, Y_2)$  to decode  $M_{12}$ . By **P3** this may be done at full rate  $\alpha \log(\kappa)$ . Similar arguments for time slots (2) and (3) demonstrate that the rates in (4)-(5) are achievable. ■

### C. Proof of Theorem 4

*Proof:* Time-sharing may again be used to achieve this region. For the converse,

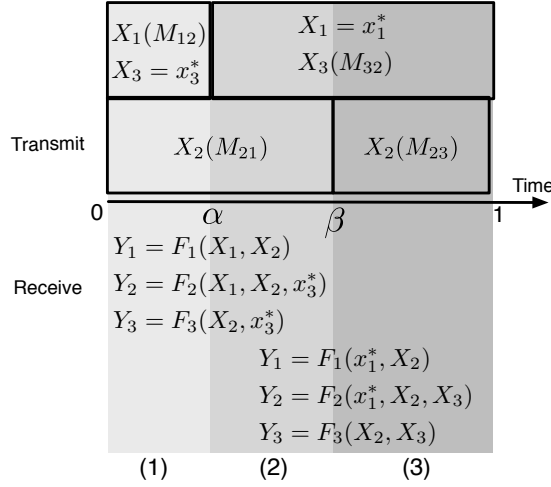


Fig. 5. Time-sharing based achievability for the proof of Theorem 2.

*Proof of bound (8):*

$$\begin{aligned}
& n(R_{12} + R_{32} + R_{34} - \epsilon) \\
& \leq I(M_{12}; Y_2^n | M_{21}, M_{23}, M_{43}) + I(M_{32}, M_{34}; Y_4^n, Y_2^n | M_{43}, M_{12}, M_{21}, M_{23}) \\
& \stackrel{(a)}{\leq} I(M_{12}; Y_2^n | M_{21}, M_{23}, M_{43}) + I(M_{32}, M_{34}; Y_2^n | M_{43}, M_{12}, M_{21}, M_{23}) \\
& \quad + I(M_{32}, M_{34}; Y_4^n | M_{43}, M_{12}, M_{21}, M_{23}, Y_2^n) \\
& \stackrel{(b)}{\leq} H(Y_2^n | M_{21}, M_{23}, M_{43}) - H(Y_2^n | M_{12}, M_{21}, M_{23}, M_{43}) + H(Y_2^n | M_{12}, M_{21}, M_{23}, M_{43}) \\
& \quad + H(Y_4^n | M_{43}, M_{12}, M_{21}, M_{23}, Y_2^n) \\
& \stackrel{(c)}{\leq} \sum_{i=1}^n [H(Y_{2,i}) + H(Y_{4,i} | M_{12}, M_{21}, M_{23}, M_{43}, Y_4^{i-1}, Y_2^n)] \\
& \stackrel{(d)}{=} \sum_{i=1}^n [H(Y_{2,i}) + H(X_{3,i} \oplus X_{4,i} | M_{12}, M_{21}, M_{23}, M_{43}, Y_4^{i-1}, X_4^i, X_3^{i-1}, Y_2^n, X_2^n)] \\
& \stackrel{(e)}{=} \sum_{i=1}^n [H(Y_{2,i}) + H(X_{3,i} | M_{12}, M_{21}, M_{23}, M_{43}, Y_4^{i-1}, X_4^i, X_3^{i-1}, X_1^n \oplus X_2^n \oplus X_3^n, X_2^n, X_1^n)] \\
& = \sum_{i=1}^n [H(Y_{2,i})] \leq n
\end{aligned}$$

where (a) follows from the chain rule. We drop two negative entropy terms in inequality (b) and notice that the second and the third entropy terms cancel each other. In (c), we apply the chain rule first, then we drop the conditioning part of the first entropy term. In (d), we construct  $X_4^i = f_4(M_{43}, Y_4^{i-1})$  and note that  $X_3^{i-1}$  may be obtained from  $Y_4^{i-1} = X_3^{i-1} \oplus X_4^{i-1}$ , given  $X_4^{i-1}$ . Adding  $X_2^n$  follows from the fact  $X_2^n = f_2(M_{21}, M_{23}, Y_2^{n-1})$ . In (e), we cancel  $X_{4,i}$  in the second entropy term since we know  $X_4^i$ . In addition, given  $M_{12}$  and  $X_2^n$ , we may construct  $X_1^n$  as illustrated in Fig. 6. Now, we may obtain  $X_3^n$  from

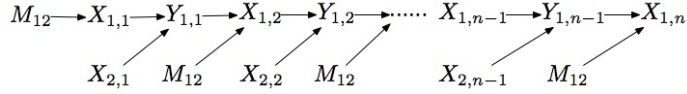


Fig. 6. The Markov chain used in the outer bound proof of Theorem 4.

$Y_2^n = X_1^n \oplus X_2^n \oplus X_3^n$ , so that the second entropy term is zero. Bound (9) follows by symmetry. ■

#### D. Proof of Theorem 6

*Proof:* We may achieve this region using two time-sharing random variables; one between nodes 1 and 3, and a second between nodes 2 and 4. For the converse,

*Proof of bound (10):*

$$\begin{aligned}
& n(R_{12} + R_{34} - \epsilon) \\
& \stackrel{(a)}{\leq} I(M_{12}; Y_2^n | M_{21}, M_{43}) + I(M_{34}; Y_4^n, Y_2^n | M_{12}, M_{21}, M_{43}) \\
& = I(M_{12}; Y_2^n | M_{21}, M_{43}) + I(M_{34}; Y_2^n | M_{21}, M_{12}, M_{43}) + I(M_{34}; Y_4^n | M_{21}, M_{12}, M_{43}, Y_2^n) \\
& \stackrel{(b)}{\leq} I(M_{12}; Y_2^n | M_{21}, M_{43}) + I(M_{34}; Y_2^n | M_{21}, M_{12}, M_{43}) + H(Y_4^n | M_{21}, M_{12}, M_{43}, Y_2^n) \\
& \stackrel{(c)}{=} I(M_{12}; Y_2^n | M_{21}, M_{43}) + I(M_{34}; Y_2^n | M_{21}, M_{12}, M_{43}) \\
& + \sum_{i=1}^n [H(X_{1,i} \oplus X_{3,i} \oplus X_{4,i} | M_{21}, M_{12}, M_{43}, Y_4^{i-1}, X_4^i, Y_2^n, X_2^n)] \\
& \stackrel{(d)}{=} I(M_{12}; Y_2^n | M_{21}, M_{43}) + I(M_{34}; Y_2^n | M_{21}, M_{12}, M_{43}) \\
& + \sum_{i=1}^n [H(X_{1,i} \oplus X_{3,i} | M_{21}, M_{12}, M_{43}, Y_4^{i-1}, X_4^i, X_1^n \oplus X_3^n, X_2^n)] \\
& = I(M_{12}; Y_2^n | M_{21}, M_{43}) + I(M_{34}; Y_2^n | M_{21}, M_{12}, M_{43}) \\
& \leq \sum_{i=1}^n [H(Y_{2,i}) - H(Y_{2,i} | Y_2^{i-1}, M_{12}, M_{21}, M_{43}) + H(Y_{2,i} | Y_2^{i-1}, M_{12}, M_{21}, M_{43})] \\
& = \sum_{i=1}^n [H(Y_{2,i})] \leq n
\end{aligned}$$

where (a) follows from Fano's inequality and the introduction of a genie  $Y_2^n$  in the second mutual information term. We drop a negative entropy term in inequality (b). In (c), we construct  $X_4^i = f_4(M_{43}, Y_4^{i-1})$  and  $X_2^n = f_2(M_{21}, Y_2^{n-1})$ . In (d), we cancel  $X_{4,i}$  in the entropy term since we know  $X_4^i$ . In addition,  $X_1^n \oplus X_3^n$  is decoded from  $Y_2^n = X_1^n \oplus X_2^n \oplus X_3^n$  since  $X_2^n$  is known. The bound (11) follows by symmetry. ■



### E. Evaluation of the sum-rate outer bound with full adaptation in Gaussian two-way IC of Theorem 11

Letting  $E[X_j X_k^*] = \lambda_{jk}$ , suppressing the subscript  $i$ , and assuming a symmetric channel, the first two terms in (26) may be bounded as

$$\begin{aligned}
& H(g_{12}X_1 + g_{32}X_3 + Z_2|X_2) - H(Z_2) \\
& \leq H(g_{12}X_1 + g_{32}X_3 + Z_2) - H(Z_2) \\
& \leq \log 2\pi e(\text{Var}(g_{12}X_1) + \text{Var}(g_{32}X_3) + 2\text{Cov}(g_{12}X_1, g_{32}X_3) + 1) - \log 2\pi e(1) \\
& = \log(\text{SNR} + \text{INR} + 2|\lambda_{13}|\cos\theta\sqrt{\text{SNR} \times \text{INR}} + 1)
\end{aligned}$$

where  $\theta$  is the angle of  $g_{12}g_{32}^*\lambda_{13}$ .

Similarly, the last two terms may be bounded as

$$\begin{aligned}
H(g_{34}X_3 + Z_4|g_{32}X_3 + Z_2, X_4) - H(Z_4) & \leq \log \left( \frac{\text{Var}(g_{34}X_3 + Z_4|g_{32}X_3 + Z_2, X_4)}{\sigma_4^2} \right) \\
& \leq \log \left( 1 + \frac{\text{SNR}(1 - |\lambda_{34}|^2)}{\text{INR}(1 - |\lambda_{34}|^2) + 1} \right). \tag{40}
\end{aligned}$$

Combining all terms, we obtain

$$\begin{aligned}
R_{\text{sym}} = \frac{R_{12} + R_{34}}{2} & \leq \frac{1}{2} \log(\text{SNR} + \text{INR} + 2|\lambda_{13}|\cos\theta\sqrt{\text{SNR} \times \text{INR}} + 1) \\
& \quad + \frac{1}{2} \log \left( 1 + \frac{\text{SNR}(1 - |\lambda_{34}|^2)}{\text{INR}(1 - |\lambda_{34}|^2) + 1} \right) \tag{41}
\end{aligned}$$

To obtain (25) one may verify that (41) is maximized at  $\lambda_{34} = 0$  and  $\lambda_{13} = 1, \theta = 0$ .

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